MATHEMATICS (9709) - AS Level

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1. Quadratics

1.1. Quadratic Formula

The quadratic formula helps us find the solutions to a quadratic equation, $ax^2+bx+c=0$.

It helps us find the values of x for which the quadratic equation equals 0, which are also called the zeros of the quadratic equation for this reason.

Quadratic formula (Given in MF19)

$$x = \frac{-(b) \pm \sqrt{(b)^2-4(a)(c)}}{2a}$$

So each of the roots of the quadratic equation $ax^2 + bx + c = 0$ are

$$x = \frac{-b + \sqrt{b^2 - 4ac}}{2a}$$

And

$$x = \frac{-b - \sqrt{b^2 - 4ac}}{2a}$$

Factorisation

Factorisation allows us to express a polynomial as the product of simpler terms.

For example:

$$x^2-2x-x+2=(x-1)(x-2)$$

Factorisation of a quadratic can be done through two methods:

- splitting the middle term
  - Re-writing $ax^2+bx+c=0$ as $ax^2+dx+ex+c=0$ [Where $d \times e = a \times c$ and $dx+ex=bx$]
  - Then factoring out the common factors in the equation
- A calculator
  - Using the "mode" button and then selecting the type of equation. This will give us the solutions to a quadratic equation, which can then be used to factorise it.

Example 1

Solve the quadratic equation $6x^2-17x+5=0$ using the quadratic formula.

Answer

Here $a=6, b=-17, c=5$. Using the quadratic formula

$$x = \frac{-(-17) \pm \sqrt{(-17)^2-4(6)(5)}}{2(6)}$$

So the 2 values of x which satisfy the equation are

$$x = \frac{5}{2} \text{ and } x = \frac{1}{3}$$

Example 2

Solve the quadratic equation $6x^2-17x+5=0$ using factorisation.

Answer

This can be factorised as

$$(2x-5)(3x-1)=0$$

- Logically, if we have $ab=0$ then either $a=0$ or $b=0$.

$$2x-5=0 \Rightarrow x=\frac{5}{2}$$

$$3x-1=0 \Rightarrow x=\frac{1}{3}$$

1.2. Completing the square

Completing the square allows us to write a quadratic, ax^2+bx+c, in the form $p(x+q)^2+r$. This is useful for a variety of reasons, which we will see later on.

- Given ax^2+bx+c, we can complete the square using the formula:

$$a\left(x+\frac{b}{2a}\right)^2+c-\frac{b^2}{4a}$$

- Comparing coefficients in the expansion of $(x+a)^2$ is another way to complete the square.

Example

Express $3x^2+9x+5$ in the form of $p(x+q)^2+r$, where p, q and r are constants.

Answer

First we identify our coefficients, $a=3, b=9, c=5$. Now we can substitute in our values into the formula:

$$3\left(x+\frac{9}{2(3)}\right)^2+5-\frac{9^2}{4(3)}$$

Which simplifies to

$$3\left(x+\frac{3}{2}\right)^2-\frac{7}{4}$$

- You can verify your answer by expanding it and comparing it with your original quadratic.

1.3. Solving quadratic equations using completing the square form

It is possible to solve quadratic equations by completing the square. Given a quadratic equation $ax^2+bx+c=0$, it can be written as $p(x-q)^2+r=0$.

- By making x the subject of the formula, we get

$$x = \pm\sqrt{\frac{-r}{p}}+q$$

- Note that if $\frac{-r}{q} < 0$, the quadratic has no real solutions.

Example

By completing the square, find the solutions to the quadratic equation $x^2-5x+6=0$.

Answer

- First we complete the square

$$x^2-5x+6 \Rightarrow \left(x-\frac{5}{2}\right)^2-\frac{1}{4}$$

$$\left(x-\frac{5}{2}\right)^2-\frac{1}{4}=0$$

- Next, we make x the subject of the formula and solve for it

$$x=\pm\sqrt{\frac{1}{4}}+\frac{5}{2}$$

$$x=3 \quad \text{or} \quad x=2$$

1.4. Finding the coordinates of the vertex

The vertex of a quadratic equation is the highest or lowest point on the curve. We can find the coordinates of the vertex after completing the square

Equation	Coordinates of vertex
$y=p(x-q)^2+r$	(q,r)
$y=p(x+q)^2-r$	$(-q,-r)$

Another way of finding coordinates of the vertex is through the general completing the square formula. Given a quadratic, $y=ax^2+bx+c$:

$$y=a\left(x+\frac{b}{2a}\right)^2+c-\frac{b^2}{4a}$$

x coordinate of vertex	$x=\frac{-b}{2a}$
y coordinate of vertex	$y=\frac{4ac-b^2}{4a}$

- The line $x=-\frac{b}{2a}$ acts as a line of symmetry for the quadratic. It splits the quadratic curve into 2 equal and mirrored parts.
- Completing the square is also called the vertex form due to this.

Example

Find the coordinates of the vertex of the quadratic $3\left(x+\frac{3}{2}\right)^2-\frac{7}{4}$.

Answer

- We observe that it has been written in the completing the square, or vertex, form.

$$3\left(x+\frac{3}{2}\right)^2-\frac{7}{4}=p(x-q)^2+r$$

$$q=-\frac{3}{2}$$

$$r=-\frac{7}{4}$$

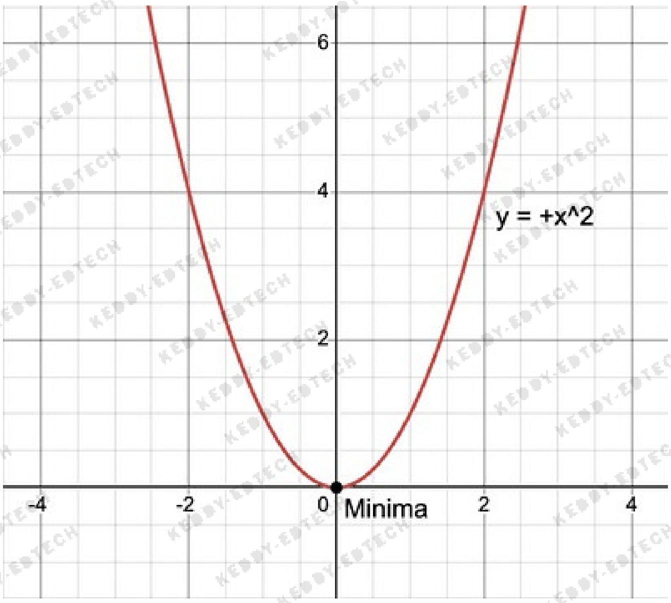
The coordinates of the vertex are $\left(-\frac{3}{2}, -\frac{7}{4}\right)$

1.5. Quadratic inequalities

An inequality question will ask us to find the values of x for which the inequality holds true.

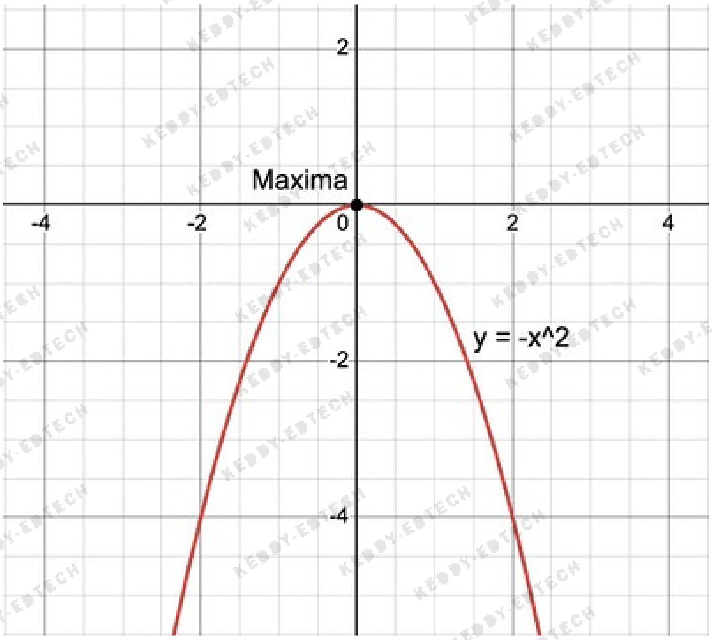
The most efficient way to solve these inequalities is to follow these steps.

- Factorise the inequality.
  - This will help us find the intersections of the quadratic with the x-axis
- Draw a simple sketch of the quadratic and label the x coordinate of the intersections.
  - The quadratic curve will have a "smiley" or an U shape if the coefficient of x^2 is positive. This point is also known as a minimum point, or a minima.



  - The quadratic curve will have a "frown" or a n shape if the coefficient of x^2 is negative.

    This point is also known as a maximum point, or a maxima.



- Using your sketch, deduce the values of x for which the inequality holds true.
- If a question asks you to find $y \le 0$ or $y \ge 0$, then the points of intersections must be included in the range of values.

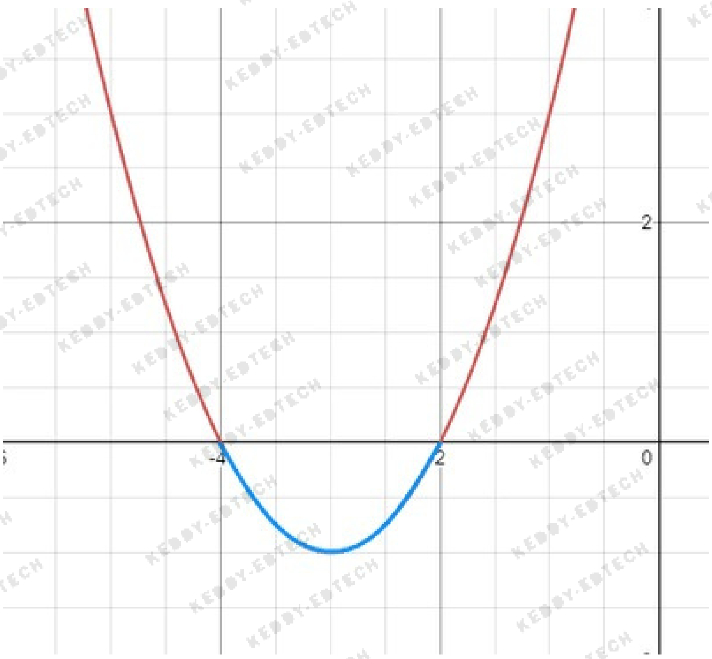
Example

Find the values of x which satisfy the inequality $x^2+6x+8<0$

Answer

- First notice that the coefficient of the quadratic is positive, $+x^2$, meaning the curve must be sketched in the shape of a smiley or an u.
- Then we can factorise x^2+6x+8 to get $(x+4)(x+2)$. Upon factoring, it is clear that the roots are $x=-2$ and $x=-4$.

We can now sketch the parabola $y=(x+4)(x+2)$, labelling the intersections with the x-axis



The blue part of the curve represents the region where $y<0$ and the red part of the curve represents the region where $y>0$

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- As we're trying to find the values of x for which $y<0$, we look at the blue region. Clearly any value that is greater than -4 and less than -2 is represented in the blue region. So the answer is $-4<x<-2$.

1.6. Discriminant of a quadratic formula

The discriminant helps us find the number of intersections of a quadratic equation with the x-axis. It can also be used to find the number of intersections of a quadratic curve with a line or another quadratic curve.

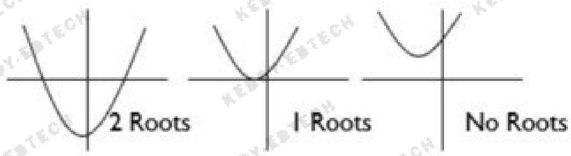
Consider b^2-4ac in quadratic equation

$$x = \frac{-b \pm \sqrt{b^2-4ac}}{2a}$$

- If $b^2-4ac>0$
  - This results in $x=\frac{-b\pm\sqrt{b^2-4ac}}{2a}$ meaning there are two distinct real roots that can be found.
- If $b^2-4ac=0$
  - As the square root is equal to 0, we get $x=\frac{-b}{2a}$, there are two equal and real roots.
- If $b^2-4ac<0$
  - There is a negative term inside the square root, later taught as a complex root, meaning there are no real solutions.

For finding the number of intersections with the x-axis, or finding the number of roots:

Discriminant	Number of roots	Nature of intersection with x-axis
$b^2-4ac>0$	Two real distinct roots	Two distinct points of intersection
$b^2-4ac=0$	Two equal real roots	One point of intersection
$b^2-4ac<0$	No real roots	No points of intersection

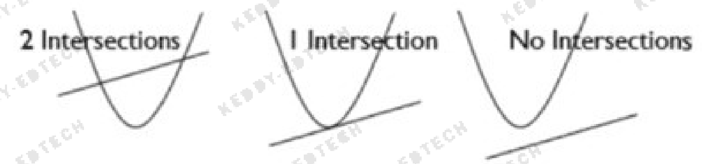


Finding the number of intersection between curves

- The theory of discriminants can be extended to find the number of intersections between a quadratic and another quadratic curve or linear line.
  - To find the number of intersections, we first equate both curves or line and then apply the discriminant formula.

  For finding the number of intersections between a quadratic and another line or curve:

Discriminant	Number of intersections	Nature of intersection with curve
$b^2 - 4ac > 0$	Two real distinct points of intersection	Line meets curve at two points
$b^2 - 4ac = 0$	One real point of intersection	Line is tangent to the curve
$b^2 - 4ac < 0$	No real points of intersections	Line does not meet the curve



Example

Given a quadratic, $kx2+4kx+3k=0$, find the range of k values for which the equation has two distinct real roots.

Answer

We need to satisfy the condition $b2-4ac>0$ to have two distinct real roots.

The coefficients of the quadratic equation can be labelled $a=k, b=4k$ and $c=3k$, leading to

$$(4\ 2k) - 4(k)(3k) > 0$$

$$16k^2 - 12k2 > 0$$

$$4k^2 > 0$$

Thus we get the range of k as

$$k > 0$$

1.7. Simultaneous equations

A simultaneous equation is a set of equations, with unknown variables, that satisfy a condition.

These can be solved through the use of the methods of elimination and substitution.

Elimination

The method of elimination requires you to multiply one, or both, equation such that one of the unknown variables shares the same coefficient in both equations.

- This may require you to multiply the equations with an integer, or an unknown variable in more complicated problems.

Example

Solve the simultaneous equations

$$4a+2b=8$$

$$a+3b=7$$

Answer

- First we make an unknown variable have the same coefficient.
  - We can do this by multiplying the second equation by 4, resulting in $4a+12b=28$, making the unknown variable a share the same coefficient in both equations.

$$4a+2b=8$$

$$4a + 12b = 28$$

- Now we can subtract both equations to get rid of one unknown variable, resulting in $2b-12b=8-28$. This will help us isolate one variable and find its value.

$$-10b = -20$$

$$b = 2$$

- Substitute the value of b into any of the two equations. This is done to find the value of the other unknown variable.

$$4a + 2(2) = 8$$

$$a = 1$$

Substitution

The second method, substitution, makes one unknown variable the subject of the equation. This is then substituted into the second equation to find its value.

- Upon finding the value of one unknown variable, we can easily find the value of the other by substituting its value into any of the two equations.

Example

Solve the simultaneous equation

$$4a+2b=8$$

$$a+3b=7$$

Answer

- Using the second equation, we make a the subject of the formula.

$$a+3b=7 \Rightarrow a=7-3b$$

- Substitute this expression into the first equation, $4a+2b=8$.

$$4(7-3b)+2b=8$$

$$2b-12b=8-28$$

$$b = 2$$

- Substitute this value of b into any of the two equations. This is done to find the value of the other unknown variable.

$$a=7-3(2)$$

$$a=1$$

1.8. Simultaneous equations with quadratics

- A simultaneous equation can include a quadratic curve and a linear line.
  - Solving these can be thought of as finding the points of intersection of both equations.

Equating

One way of solving simultaneous equations is by equating both curves, or curve and line. This will result in a quadratic equation which can be solved to find the points of intersection.

Example

Find the coordinates of the intersections of the curve $y=4x^2-7$ and line $x+y=5$ by equating both equations.

Answer

- Make y the subject of the formula in the line equation, and square both sides to get an equation in terms of y^2.

$$y=5-x \Rightarrow y^2=(5-x)^2$$

- Now we can equate both equations, as their points of intersections must have the same y coordinate.

$$(5-x)^2=4x^2-7$$

$$x^2-10x+25=4x^2-7$$

$$0=3x^2+10x-32$$

We can solve this quadratic equation in terms of x, to get the x coordinates of intersection.

$$3x^2+10x-32 \Rightarrow (3x+16)(x-2)$$

$$x=-\frac{16}{3} \text{ or } x=2$$

- Substitute this value of x into the linear equation to find the y coordinates of intersection.

$$-\frac{16}{3}+y=5 \Rightarrow y=\frac{31}{3}$$

$$2+y=5 \Rightarrow y=3$$

- The solutions to this simultaneous equation, or points of intersection of both equations, is:

$$\left(-\frac{16}{3},\frac{31}{3}\right) \text{ and } (2,3)$$

Substitution

Another way of solving simultaneous equations is through the use of a substitution. This allows us to write an equation of one curve in terms of x or y. The new expression can then be substituted into the second equation to get the points of intersection.

Example

Find the coordinates of the intersections of the curve $y^2=4x^2-7$ and line $x+y=5$ using a substitution.

Answer

- Make x the subject of the formula in the line equation to get $x=5-y$. Now we can substitute this expression as x in the quadratic equation.

$$y^2=4(5-y)^2-7$$

$$y^2=100-40y+4y^2-7$$

$$3y^2-40y+93=0$$

We can solve this quadratic equation in terms of y, to get the y coordinates of intersection.

$$3y^2-40y+93 \Rightarrow (3y-31)(y-3)$$

$$y=\frac{31}{3} \text{ or } y=3$$

- Substitute this value of y into the linear equation to find the x coordinates of intersection.

$$x+\frac{31}{3}=5 \Rightarrow x=-\frac{16}{3}$$

$$x+3=5 \Rightarrow x=2$$

- The solutions to this simultaneous equation, or points of intersection of both equations, is:

$$\left(-\frac{16}{3},\frac{31}{3}\right) \text{ and } (2,3)$$

1.9. Substitutions

Many questions may ask you to solve unique equations, which may not resemble a quadratic at first glance. These can be solved by making a clever substitution, by the use of a variable, and then factorising it to solve for the values of x.

The table below shows an example of possible substitutions that one can use

Equation	Substitution
ax^4+bx^2+c	$u=x^2$ and $u^2=x^4$
ax^6+bx^3+c	$u=x^3$ and $u^2=x^6$
$ax+b\sqrt{x}+c$	$u=\sqrt{x}$ and $u^2=x$
$ax^{\frac{2}{3}}+bx^{\frac{1}{3}}+c$	$u=x^{\frac{1}{3}}$ and $u^2=x^{\frac{2}{3}}$
$ax^{-2}+bx^{-1}+c$	$u=x^{-1}$ and $u^2=x^{-2}$

Example 1

Solve the equation $x^4-5x^2+4=0$

Answer

Let $u = x^2$, such that $u^2 = x^4$ and $x = \pm\sqrt{u}$

Using this substitution, we get $u^2 - 5u + 4 = 0$. This quadratic equation can be factorised easily.

$$(u-4)(u-1)=0$$

$$u = 4 \Rightarrow x^2 = 4 \Rightarrow x = \pm 2$$

$$u = 1 \Rightarrow x = 1^2 \Rightarrow x = \pm 1$$

Example 2

Solve the equation $6x + \sqrt{x} - 1 = 0$

Answer

Let $u = \sqrt{x}$, such that $u^2 = x$ and $x = u^2$

Using this substitution, we get $6u^2 + u - 1 = 0$. This quadratic can be factorised easily.

$$(3u - 1)(2u + 1) = 0$$

$$u = \frac{1}{3} \Rightarrow \sqrt{x} = \frac{1}{3} \Rightarrow x = \frac{1}{9}$$

$$u = \frac{-1}{2} \Rightarrow \sqrt{x} = \frac{-1}{2} \Rightarrow x = \frac{1}{4}$$

1. Functions

- Afunctionassigns a collection of *x* values to only one *y* value.
    - A function can be denoted by *f*(*x*), *g*(*x*), *h*(*x*) etc.
- Inputting a *x* value into the function *f* outputs a *y* value [also written as *f*(*x*)]

For example

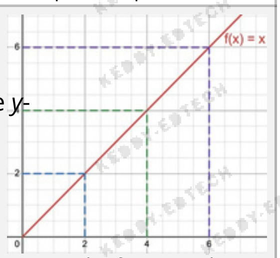
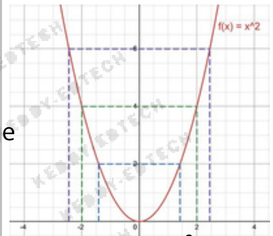
$f(x) = x2 \ + 4x + 4$

When $x = 2$

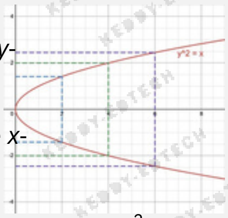
$f(2) = (2)2 + 4(2) + 4 = 16$

1.1. Types of functions and relations

There are many types of relations a function can have:

Function	Definition	Graphical representation
One to One	One *x*-value input gives one *y*-value output. As an example, all linear functions of the type *f*(*x*)=*mx*+*c* are One to One.	 Graph of $y = x$, also written as $f(x) = x$.
Many to One	Two *x*-value inputs give the same *y*-value output.	 Graph of $y = x^2$, also written as $f(x) = x2$.

Another type of relation, that is not a function is:

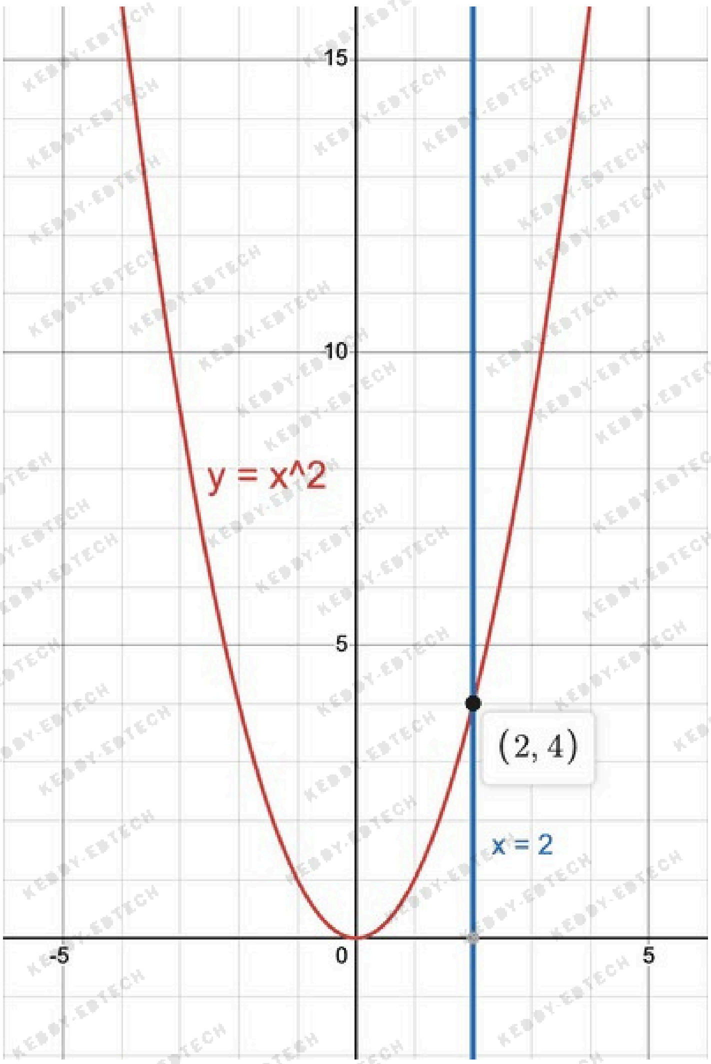
Relation	Definition	Graphical representation
One to many [Not a function]	One *x*-value input gives two *y*-value outputs. This is not a function as it gives us more than one *y*-value output with one *x*-value input, contradicting the definition of a function.	 Graph of $y^2 = x$.

Vertical Line Test to check for Existence of a Function

To check for the existence of a function, we make use of the vertical line test.

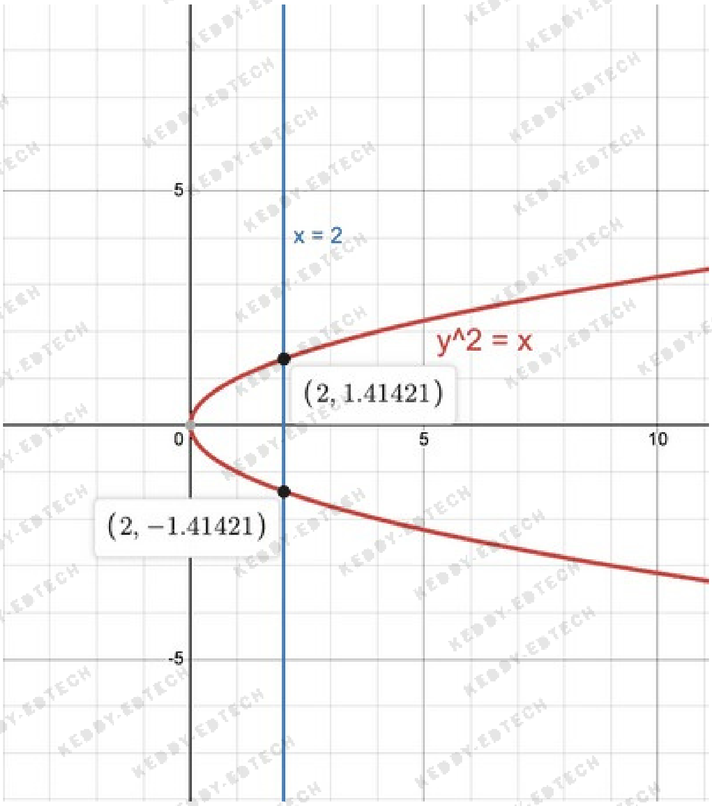
- If a vertical line, *x*=*a*, cuts the the function at one point only then the function exists
    - Note that *a* is a constant which is included within the interval of *x*-values the function is defined for.

Curve passes the vertical line test



- As the curve *y*=*x*2 passes the vertical line test, it is a function.

Curve fails the vertical line test



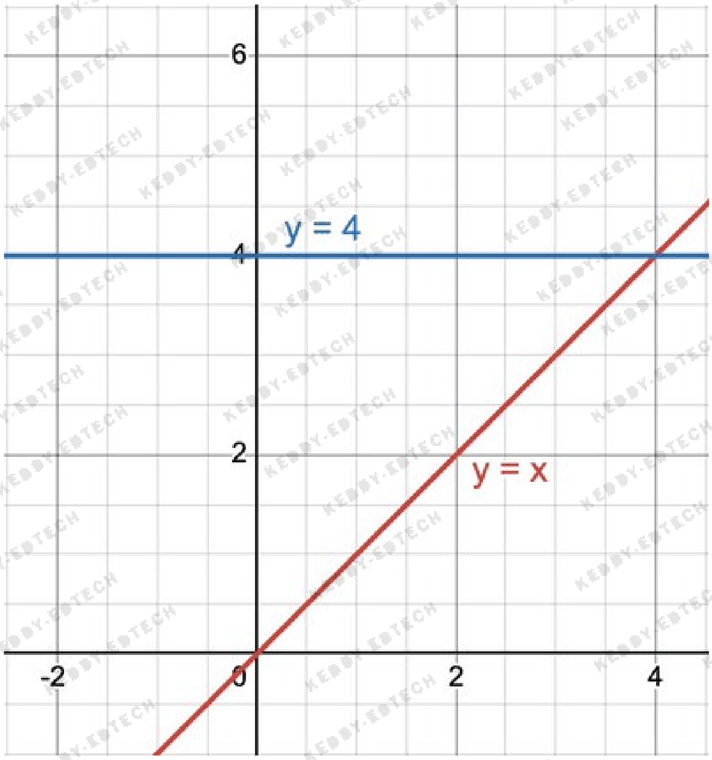
- As the curve $y^2 = x$ fails the vertical line test, it is not a function.

Horizontal Line Test to check for One to One Relation

To check if a function has a One to One relation, we make use of the horizontal line test.

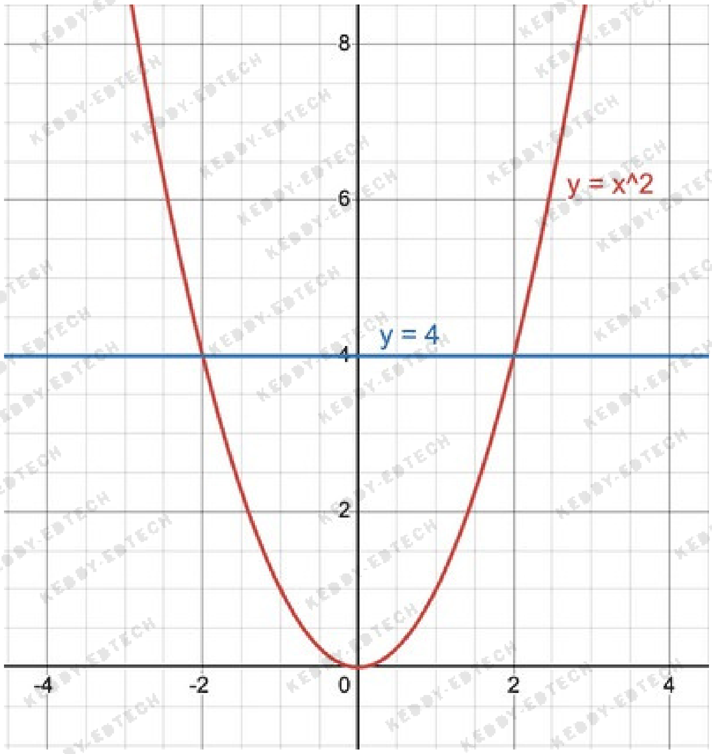
- If a horizontal line, $y=a$, cuts the function at one point only then the function has a One to One relation. Else it will have a Many to One relation.
    - Note that a is a constant which is included within the interval of y-values the function is defined for.

Curve passes the horizontal line test



- As the curve $y=x$ passes the horizontal line test, the function has a One to One relation.

Curve fails the horizontal line test



- As the curve $y=x^2$ fails the horizontal line test, the function does not have a One to One relation. It instead has a Many to One relation in this case.

1.2. Domain and Range

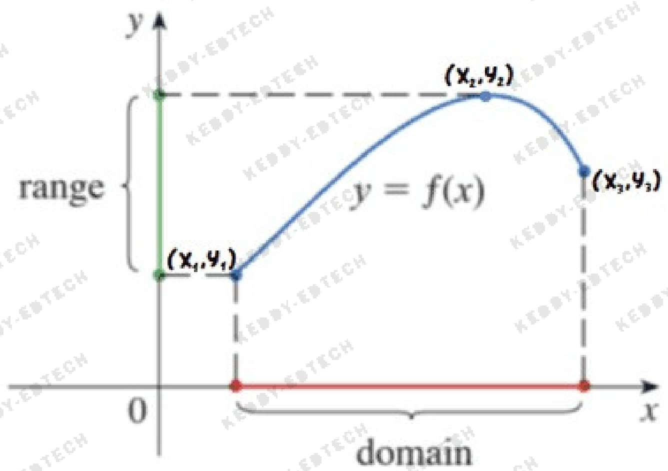
Domain

- The domain of a function is the set of *x*-values, or inputs, that the function *f* is defined for.
    - It consists of the smallest and biggest *x*-value the function accepts as an input, or is defined for.
    - Example: $x \in R$ implies the function the function is defined for all real values of *x*.
    - $4 < x < 10$ implies the function is defined between $x = 4$ and $x = 10$.

Range

- The range of a function is the set of corresponding *y*-values, or outputs, that the function *f* is defined for.
    - It consists of the lowest and the highest *y*-value the function outputs for a given domain.
    - Example: $y \in R$ implies the function is defined for all real values of *y*.
    - $-2 \le y \le 5$ implies the function is defined between $y = -2$ and $y = 5$, where both values are inclusive.

Example



For this curve, $y = f(x)$:

- The range is $y_1 \le f(x) \le y_2$
    - Also written as $y_1 \le y \le y_2$
- The domain is $x_1 \le x \le x_3$

1.3. Domain and Range of a Quadratic

Domain of a Quadratic

- The domain of a quadratic is $x \in R$, unless it is restricted.
- The line $x = -\frac{b}{2a}$, or $x = q$ (where *q* is the *x* coordinate of the vertex), is the line of symmetry
    - We can find a restricted domain such that the function has a One to One relation.

Range of a Quadratic

- The range of a quadratic depends on its minimum or maximum *y* value.

Consider a quadratic equation written in the form

$$ax^2 + bx + c = 0$$

We can complete the square for this equation to get the *y*-coordinate of the vertex, or the minimum or maximum *y* value.

Equation	Coordinates of vertex	Nature of vertex	Range
$y = p(x - q)^2 + r$	(q, r)	Maximum point ($a < 0$ or $p < 0$)	$r \ge y$ or $r > y$
$y = p(x - q)^2 + r$	(q, r)	Minimum point ($a > 0$ or $p > 0$)	$r \le y$ or $r < y$

- Note that the type of inequality sign you use (> or ≤) depends on if the domain includes the *x* coordinate of the vertex, *q*, using ≤ and ≥.

Example 1

Given a quadratic, $f(x) = 2x^2 - 2x + 3$ with a domain of $a \le x$, find the smallest value of *a* for which the function has a One to One relation.

Answer

First we complete the square for *f*(*x*)

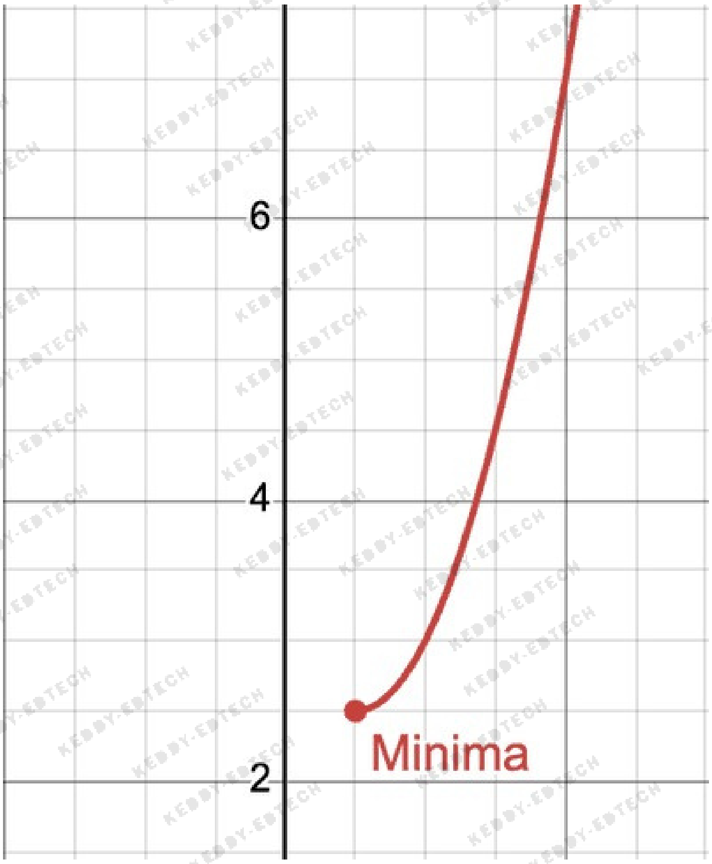
$$2\left(x - \frac{1}{2}\right)^2 + \frac{5}{2}$$

As $a > 0$ and $p > 0$, the vertex is a minimum point with coordinates $\left(\frac{1}{2}, \frac{5}{2}\right)$.

So the smallest value of *a*, for which the function has a One to One relation, is $\frac{1}{2}$ and the domain is

$$\frac{1}{2} \le x$$

The graph is shown below



Example 2

Find the range of $f(x)=-2x^2+x+4$, where $x \in \mathbb{R}$.

Answer

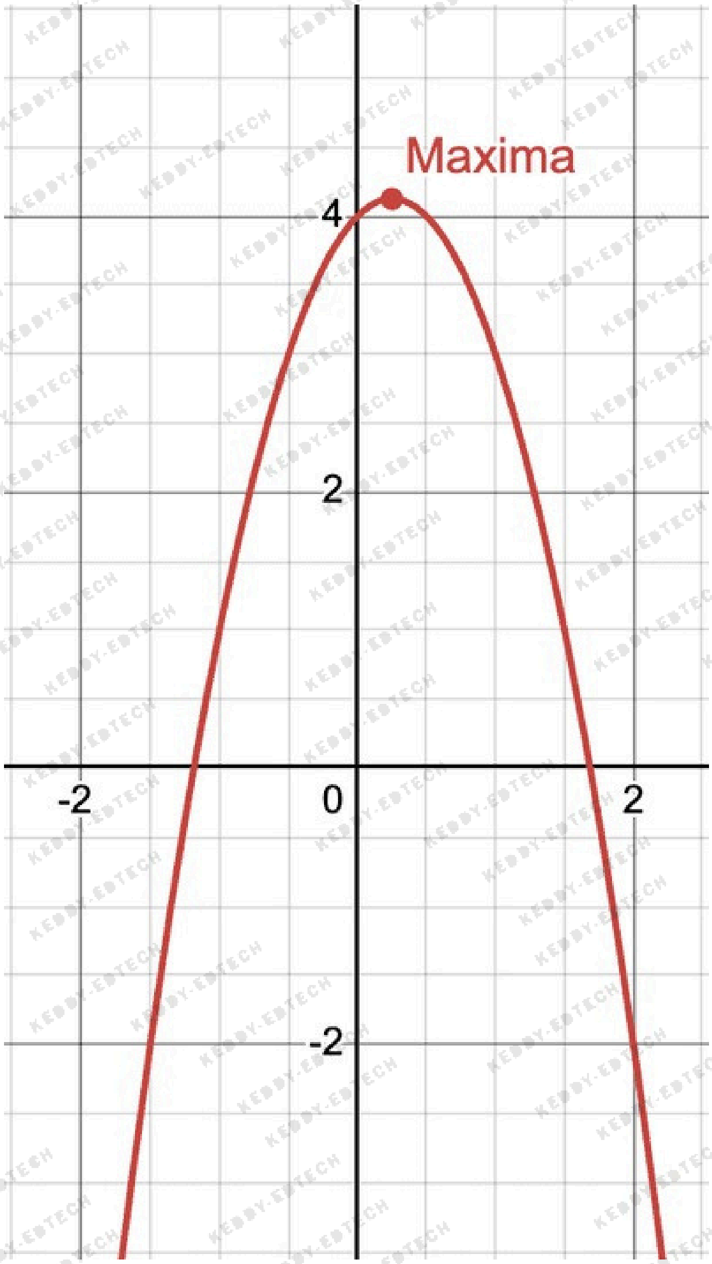
First we complete the square for $f(x)$

$$f(x) = -2\left(x-\frac{1}{4}\right)^2+\frac{33}{8}$$

As $a<0$ and $p<0$, the vertex is a maximum point. We can now find the range as

$$\frac{33}{8} \geq y$$

The graph is shown below



1.4. Different types of Functions

Square root function

- Consider a function $f(x)=\sqrt{ax+b}$
  - The domain must be chosen such that $ax+b>0$ always. In this case the domain is $x>-\frac{b}{a}$.
  - The range of such a function is always $y>0$
- For a more general way of solving, consider a function $f(x)=\sqrt{g(x)}$ [where $g(x)$ is any function such as a linear function, quadratic, or even a higher degree polynomial]
  - The domain must be chosen such that $g(x)\geq 0$ always
  - The range of function is always a positive, non-zero, interval.

Rational function

- A rational function is a function written in the form $\frac{f(x)}{g(x)}$, where $g(x) \neq 0$.

- To find the domain
  - Find the values of x for which $g(x)=0$. These values of x cannot be inputted into the function and must be excluded in the domain.
- To find the range

Input a large x value, such as 999 and −999. This will help you find the y value which the rational function is approaching, but never reaches. This limits the range of the rational function.

If the function is written in the form

$$f(x)=a+\frac{g(x)}{h(x)} \quad \text{for } x\in\mathbb{R}$$

Where a is a non-zero constant, and $g(x)$ and $h(x)$ are functions.

Then the range is $f(x)<a$ or $f(x)>a$ depending on if the function is approaching the value a from above or below the line $y=a$. This can be checked by inputting a large x value and seeing if the output is greater than or less than a.

- Note that the inequality can change if the domain changes.

Example

Find the range of

$$f(x)=\frac{2}{3}+\frac{2}{3(3x-1)} \quad \text{for } x\neq\frac{1}{3}$$

Answer

Inputting in a large x value, such as 999, gives us

$$f(999)=\frac{2}{3}+\frac{2}{3(3(999)-1)}$$

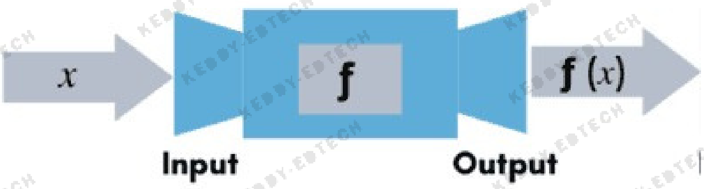
Which is greater than $\frac{2}{3}$, so the function is approaching this value from above. This gives us the range of the function as

$$y>\frac{2}{3}$$

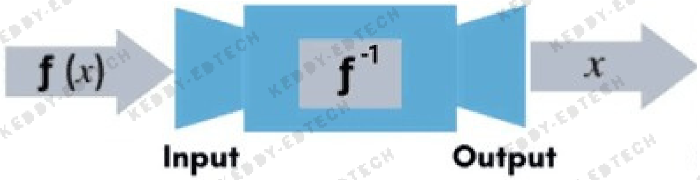
1.5. Inverse Function

- An inverse function reverses another function.
  - Commonly denoted by $f^{-1}(x)$.
  - It is also important to note that $f^{-1}(x)$ does not imply $\frac{1}{f(x)}$.
- Inputting x into the function f outputs $f(x)$ while inputting $f(x)$ into f^{-1} outputs x [also written as $f^{-1}(x)$]

Consider a function f and its inverse function f^{-1}. These two functions can be represented as



And



This gives us an important result:

$$f^{-1}(f(x))=f(f^{-1}(x))=x$$

- An inverse function, f^{-1}, can only exist if the function f has a One to One relation.
  - To check if an inverse function exists, we can use the horizontal line test.

Domain and Range of an Inverse Function

Using the previous result, $f^{-1}(f(x))=f(f^{-1}(x))=x$, we can find the domain and range of an inverse function:

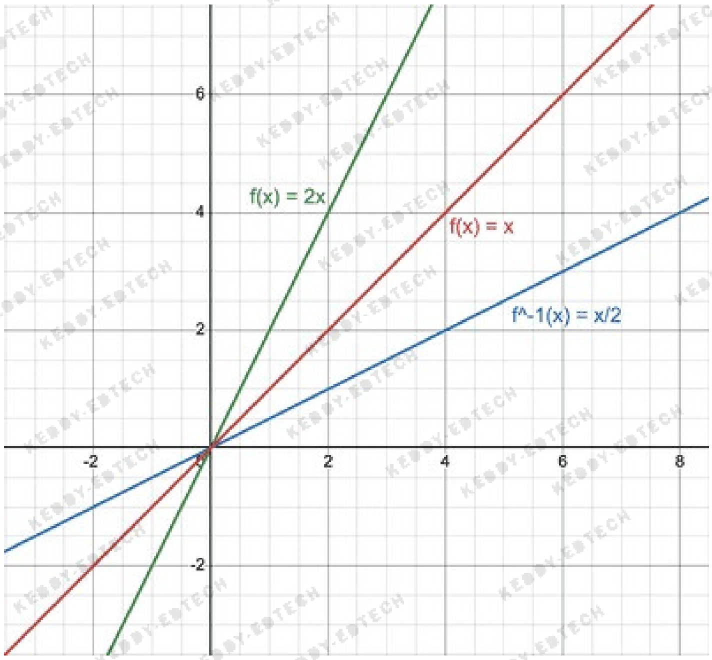
Function	Domain	Range
$f(x)$	$a<x<b$	$c<y<d$
$f^{-1}(x)$	$c<x<d$	$a<y<b$

We observe:

- The domain of $f(x)$ is the range of $f^{-1}(x)$, and vice versa
- The range of $f(x)$ is the domain of $f^{-1}(x)$, and vice versa.

Relationship between a Function and its Inverse

Using the domain and range of a function and its inverse, we can graph both functions



Observations:

- Both functions are symmetric to the line $f(x)=x$
- Both functions only intersect at $f(x)=x$

Finding an Inverse Function

- We can find the equation for an inverse function by "swapping" all the x variables to y, and y variables to x, and then making y the subject of the formula again.

Example

Given $f(x)=2x+3$, find the inverse of this function.

Answer

We start by swapping the x and y values

$$y=2x+3 \Rightarrow x=2y+3$$

Now we can make y the subject of the formula

$$y=\frac{x-3}{2}$$

So we get the inverse function as

$$f^{-1}(x)=\frac{x-3}{2}$$

Example 1

Given

$$f(x)=(x-3)^2-2 \quad \text{for } x>4$$

Find an expression for $f^{-1}(x)$ and find the domain of f^{-1}.

Answer

We start by swapping the x and y values.

$$y=(x-3)^2-2 \Rightarrow x=(y-3)^2-2$$

Now we can make y the subject of the formula

$$y=\pm\sqrt{x+2}+3$$

- To correctly pick between + and - for the square root, we look at the domain of f. As the domain of f is $x>4$, the range of $f-1$ must be $y>4$. Due to this, we need to pick the positive value of the square root, as it is the only way we will get a y value output greater than 4.

So we get the inverse function as

$$f^{-1}(x)=+\sqrt{x+2}+3$$

We can find the domain using the fact that the range of f is the domain of $f-1$.

At $x=4$

$$f(4)=(4-3)^2-2=-1$$

At $x=5$

$$f(5)=(5-3)^2-2=2$$

The y value ouput of the quadratic increases as the x value input increases as well. So the range of f is

$$f(x)>-1$$

This gives us the domain of f^{-1} as

$$x>-1$$

Example 2

Given

$$f(x)=-(x-3)^2+4 \quad \text{for } x\geq m$$

Find the smallest value of m such that the inverse function f^{-1} exists.

Answer

- An inverse function, $f-1$, can only exist if the function f has a One to One relation.

So we need to find the value of m such that $f(x)$ is an One to One function.

We can deduce that the coordinates of the vertex are (3,4) from the given equation. Using this, we get the smallest value of m as 3.

So the domain for which the inverse function exists is

$$x\geq 3$$

1.6. Composite Function

- A composite function is a combination of two functions
  - Commonly denoted by $g(f(x))$ or $gf(x)$.
- Inputting a x value into the function f outputs $f(x)$. The same value, $f(x)$, is then inputted into another function g which outputs $g(x)$.

A summary of the process has been described in the image

- It is also important to note that $g(f(x))$ does not always equal $f(g(x))$
- $f2(x)$ implies $f(f(x))$, unless stated otherwise.

Domain and Range of a Composite Function

- As $f(x)$, or the y-value output of the function f, is inputted into the function g as a x-value input
  - The domain and range of g depends on the range of f.
- If the interval of range values for $f(x)$ is not included in the interval of domain values for $g(x)$, the composite function cannot exist.
  - If the range of $f(x)$ is included in the domain of $g(x)$, but is a smaller interval, then the domain of the composite function becomes a smaller interval too.

Example 1

Given

$$f(x) = 2(x+3)^2 - 7 \quad \text{for } x \le -4$$

$$g(x) = 2x - 3 \quad \text{for } x \le k$$

Find the largest value of k for which the composite function $f(g(x))$ is defined.

Answer

- For a composite function $f(g(x))$ to exist, the range of $g(x)$ must be included in the domain of $f(x)$.

The domain of $f(x)$ is $x \le -4$, so we must find a value of k for which the range of $g(x)$ is $y \le -4$

Letting $g(x) = -4$, we get

$$2x - 3 = -4 \Rightarrow x = -\frac{1}{2}$$

So the largest value of k is

$$k = -\frac{1}{2}$$

Example 2

Given

$$f(x) = -2x^2 + 12x - 3 \quad \text{for } x \in \mathbb{R}$$

$$g(x) = 2x + 5 \quad \text{for } x \in \mathbb{R}$$

Find the values of x for which $g(f(x)) + 1 = 0$.

Answer

We can find the function $g(f(x))$ by inputting $f(x)$ into $g(x)$

$$g(f(x)) = 2(-2x^2 + 12x - 3) + 5 \Rightarrow -4x^2 + 24x - 1$$

Now we can input this value into the equation we need to solve, $f(g(x)) + 1 = 0$

$$-4x^2 + 24x - 1 + 1 = 0 \Rightarrow 4x(-x + 6) = 0$$

So the values of x which satisfy the equation are

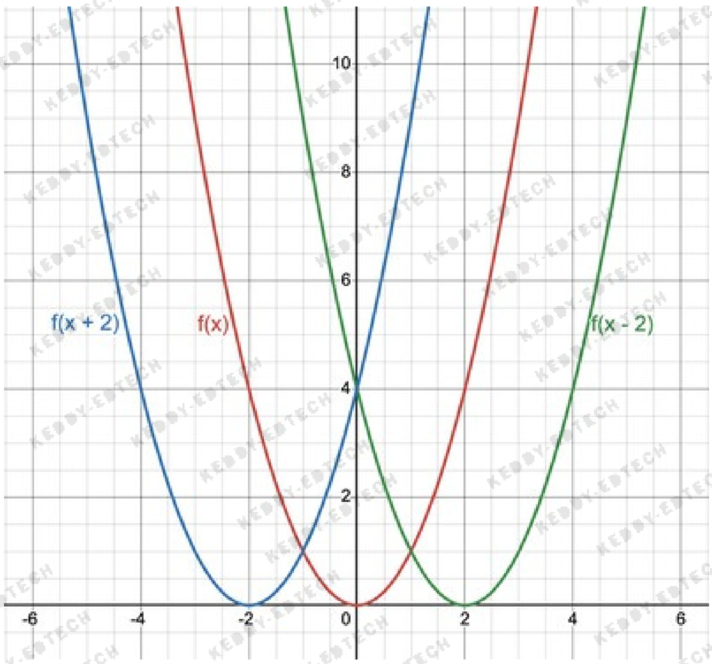
$$x = 0 \text{ or } x = 6$$

1.7. Translations

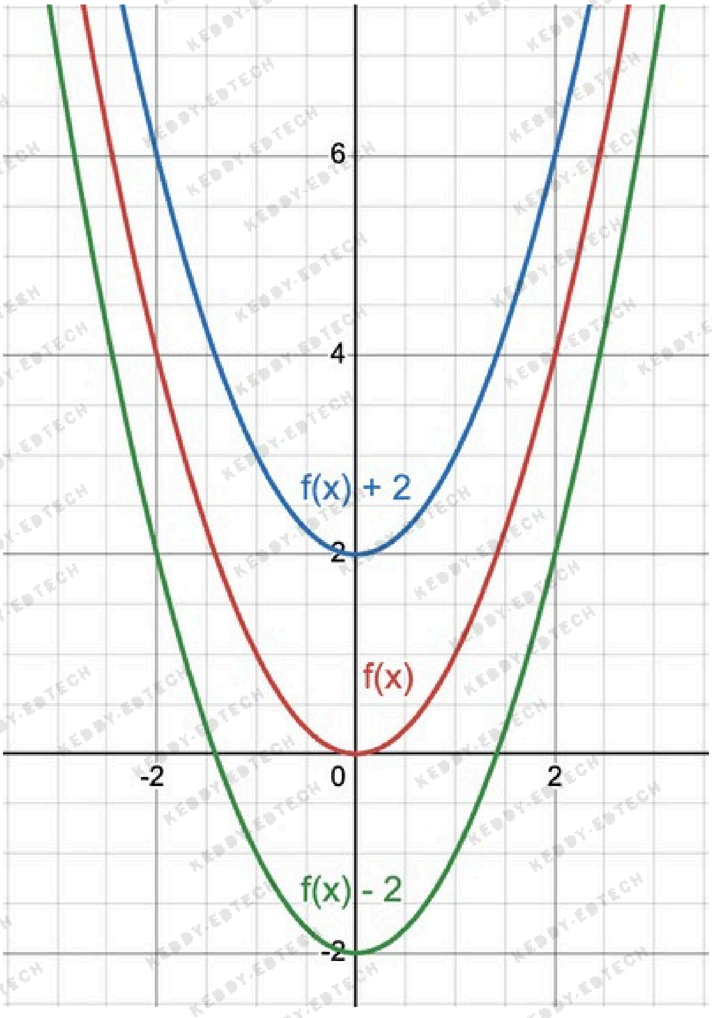
- A translation moves all the points on a curve in the x or y direction, depending on the translation.

Function	Transformation	Result	Geometric interpretation	New points
$f(x)$	$(a, 0)$	$f(x - a)$	Translation of a units in the rightward, or positive, x direction.	$(x + a, y)$
$f(x)$	$(-a, 0)$	$f(x + a)$	Translation of a units in the leftward, or negative, x direction.	$(x - a, y)$
$f(x)$	$(0, b)$	$f(x) + b$	Translation of b units in the upwards, or positive, y direction.	$(x, y + b)$
$f(x)$	$(0, -b)$	$f(x) - b$	Translation of b units in the downwards, or negative, y direction.	$(x, y - b)$

x direction translation



y direction translation



Example

The graph of $f(x)=x2+3x+2$ has been translated by the vector (-13) . Find the equation of the resulting graph.

Answer

Applying the translation to the function $f(x)$, we get:

$$y=f(x+1)+3$$

$$y=(x+1)2+3(x+1)+2+3$$

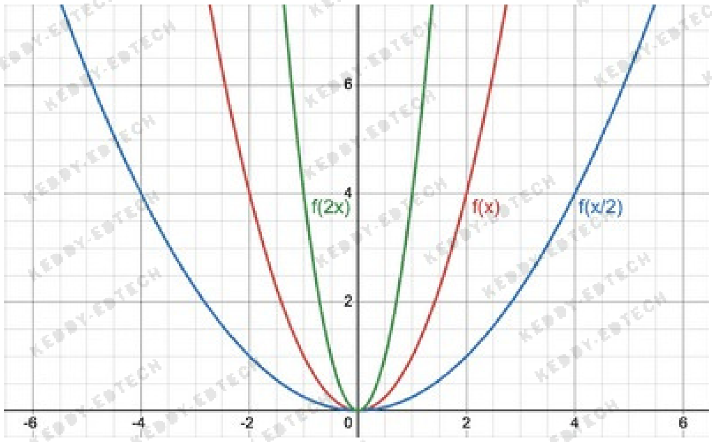
$$y = x^2+5x+9$$

1.8. Stretches

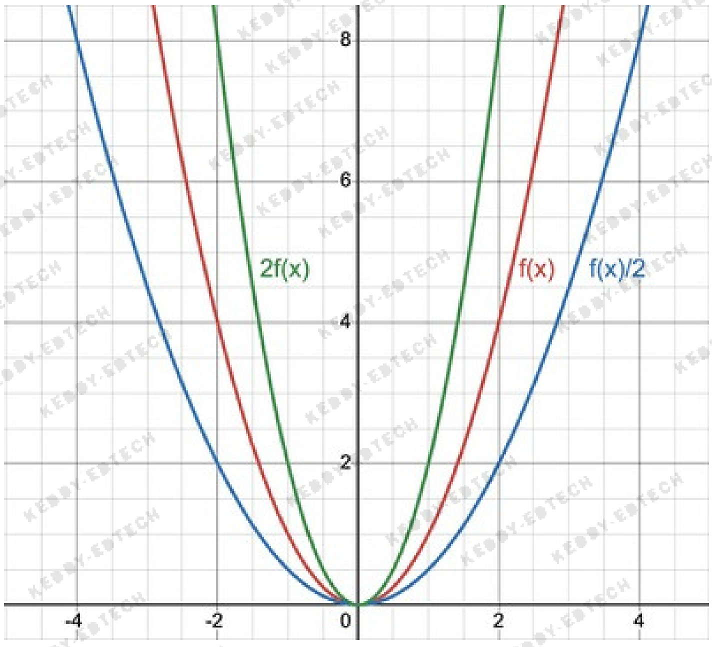
- A stretch multiplies all the x or y points on a curve by the stretch factor a.

Function	Stretch factor and direction	Result	Geometric interpretation	New points
$f(x)$	Stretch in the x-axis by factor of $\frac{1}{a}$	$f(ax)$	Stretch parallel to the x-axis by a scale factor of $\frac{1}{a}$	$(\frac{x}{a}, y)$
$f(x)$	Stretch in the x-axis by factor of a	$f(\frac{x}{a})$	Stretch parallel to the x-axis by a scale factor of a	(ax, y)
$f(x)$	Stretch in the y-axis by a factor of $\frac{1}{a}$	$\frac{f(x)}{a}$	Stretch parallel to the y-axis by a scale factor of $\frac{1}{a}$	$(x, \frac{y}{a})$
$f(x)$	Stretch in the y-axis by a factor of a	$af(x)$	Stretch parallel to the y-axis by a scale factor of a	(x, ay)

Stretch parallel to x-axis



Stretch parallel to y-axis



Example 1

The graph of $f(x)=5x2+2$ is stretched by a factor of 2 along the y-axis, find the resulting equation of the graph.

Answer

for this stretch factor

$$y = 2f(x)$$

$$y = 2(5x^2+2)$$

$$y = 10x^2+4$$

Example 2

The graph of $y=x^2+3x+2$ is stretched by a factor of $\frac{1}{3}$ along the x-axis, find the resulting equation of the graph.

Answer

For this stretch factor

$$y = f(3x)$$

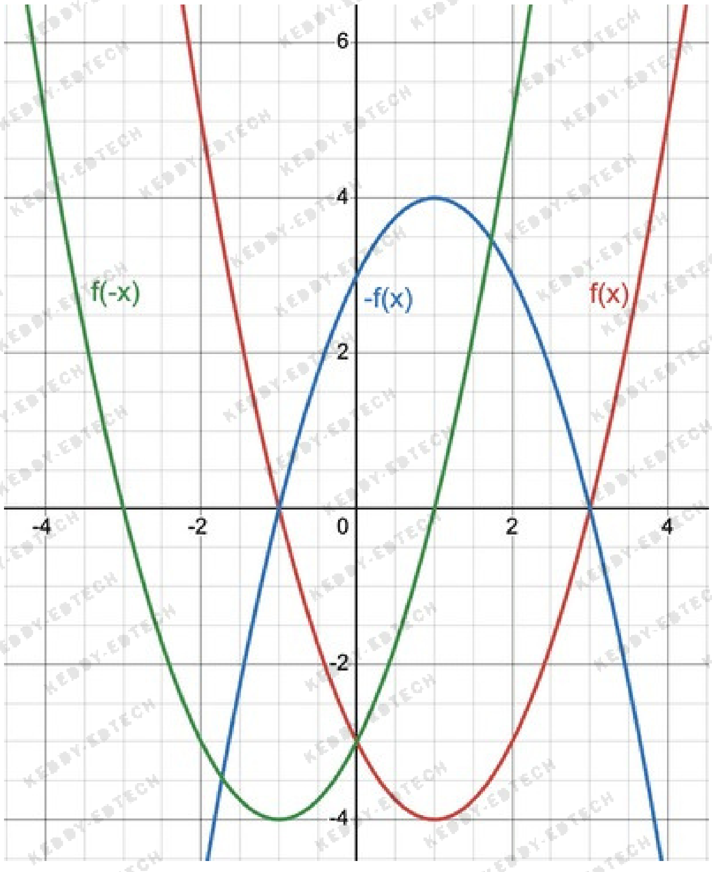
$$y = (3x)^2 + 3(3x) + 2$$

$$y = 9x^2+9x+2$$

1.9. Reflections

- A reflection transformation reflects the graph to the x or y axis.

Function	Reflection	Result	Geometric interpretation	New points
$f(x)$	Reflection in the x-axis	$-f(x)$	Reflection in the x-axis. Graph is mirrored in the x-axis, which also acts as a line of symmetry to $f(x)$ and $-f(x)$.	$(x,-y)$
$f(x)$	Reflection in the y-axis	$f(-x)$	Reflection in the y-axis. Graph is mirrored in the y-axis, which also acts as a line of symmetry to $f(x)$ and $f(-x)$	$(-x, y)$



Example

Given the equation $f(x) = 2x^2+3x+2$, find the resulting equations after a x and y axis reflection.

Answer

For a reflection in the y-axis

$$y = f(-x)$$

$$y = 2(-x)^2 + 3(-x) + 2$$

$$y = 2x^2-3x+2$$

For a reflection in the x-axis

$$y = -f(x)$$

$$y = -(2x^2+3x+2)$$

$$y = -2x^2-3x-2$$

1.10. Order of Transformations

Order
Translation parallel to x-axis
Stretch parallel to x-axis
Reflection in the y-axis
Stretch parallel in y-axis
Reflection in the x-axis
Translation parallel to y-axis

- To find out the transformations applied to a quadratic
  - Complete the square and then compare it with *f*(*x*)

Example

Functions *f* and *g* are both defined for $x \in \mathbb{R}$ and are given by

$$f(x) = (x - 2)2 + 5$$

$$g(x) = 2[(x + 1)2 + 5]$$

Describe fully the two transformations that have been combined to transform *f*(*x*) to *g*(*x*) and in the order they occur in.

Answer

The first transformation is the translation in the *x*-axis by 3 in the rightward direction. This can also be written as the vector (-30)

$$f(x+3)=((x+3)-2)2+5 \Rightarrow f(x+3)=(x+1)2+5$$

The second transformation is a stretch parallel to the *y*-axis by a stretch factor of 2.

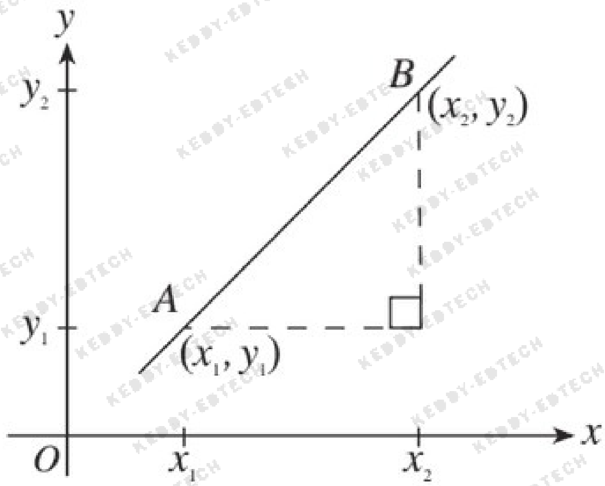
$$2f(x + 3) = 2[(x - 2)2 + 5]$$

1. Coordinate Geometry

1.1. Distance formula

- The distance formula helps us find the distance, or length, between two points.

Let us consider two points, on the same linear line, $A(x_1, y_1)$ and $B(x_2, y_2)$. The lines $x = x_2$ and $y = y_1$ can be drawn, resulting in a right angled triangle with the line AB as the hypotenuse.



The length of the base is obtained from $x_2 - x_1$ and the height is obtained from $y_2 - y_1$.

- Using Pythagoras theorem, we can derive the distance formula for two points, A and B, as:

$$AB = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$$

Mid-point formula

- The mid-point formula helps us find a point, usually denoted by M, which is equidistant from two other points.The mid-point of two points can be found using:

$$M\left(\frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2}\right)$$

Example 1

The distance between points $A(0,4)$ and $B(a,1)$ is 5 units. Find the value of the positive integer a.

Answer

Using the distance formula:

$$AB = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$$

We can substitute our values in

$$5^2 = (a - 0)^2 + (1 - 4)^2$$

$$a^2 = 16 \Rightarrow a = 4$$

Example 2

A line segment, joining points $A(2,-3)$ and $B(4,a)$ has a midpoint $M(b,-2)$. Find the values of a and b.

Answer

Using the mid-point formula

$$M = \left(\frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2}\right)$$

The y value of the mid-point can be represent as

$$y_M = \frac{y_1 + y_2}{2}$$

$$-2 = \frac{-3 + a}{2}$$

$$a = -1$$

The x value of the mid-point can be represented as

$$x_M = \frac{x_1 + x_2}{2}$$

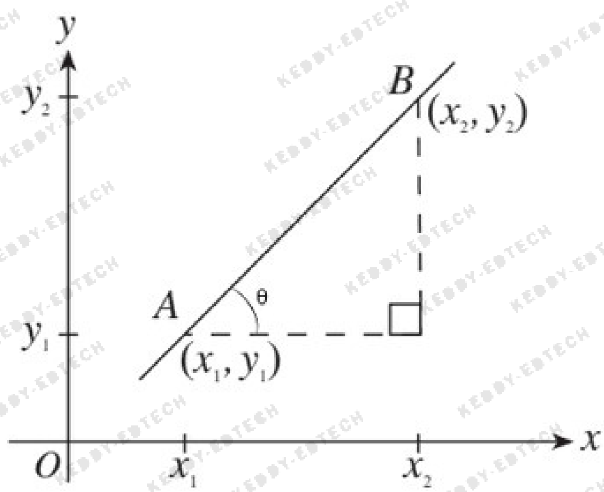
$$b = \frac{2 + 4}{2}$$

$$b = 3$$

1.2. The Gradient

- The gradient, also called the slope, describes the inclination of a line. It gives a numerical value to the angle a line makes with the positive x-axis.
  - It is popularly denoted by m in the equation $y = mx + c$.

Consider two points, $A(x_1, y_1)$ and $B(x_2, y_2)$. A right angled triangle can be drawn, with a similar explanation to the one in the distance formula, with the hypotenuse line AB.



We can apply the trigonometric ratio

$$tan(\theta) = \frac{\text{length of opposite side}}{\text{length of adjacent side}}$$

In the context of the diagram above

$$tan(\theta) = \frac{y2 - y1}{x2 - x1}$$

We will also denote m as

$$m = \frac{y2 - y1}{x2 - x1}$$

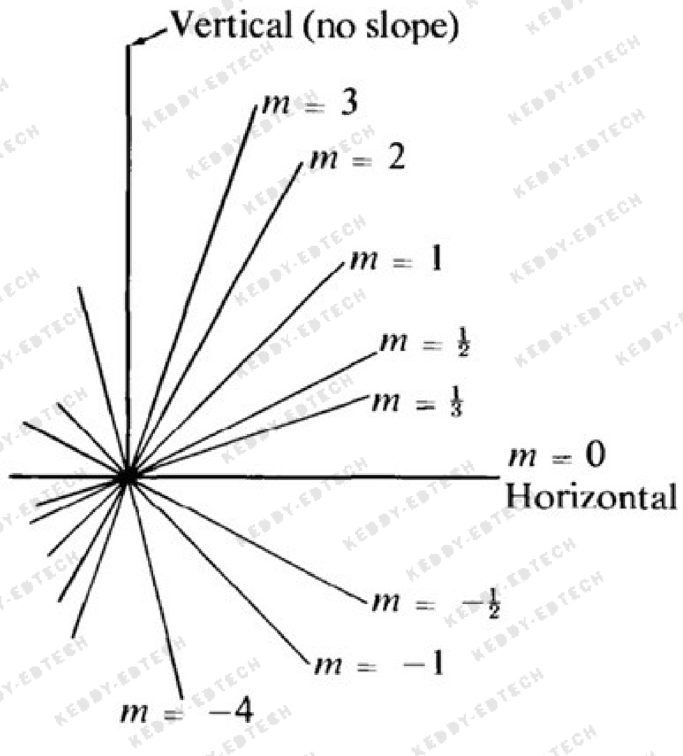
Such that we are able to find the angle of inclination, θ, by:

$$tan(\theta) = m$$

$$\vartheta = tan^{-1}(m)$$

Magnitude of the Gradient

- The function is going to have an increasing *y* value, as the *x* value increases, if the gradient has a positive value. This means the curve is moving upwards.
- The function is going to have a decreasing *y* value, as the *x* value increases, if the gradient has a negative value. This means the curve is moving downwards.



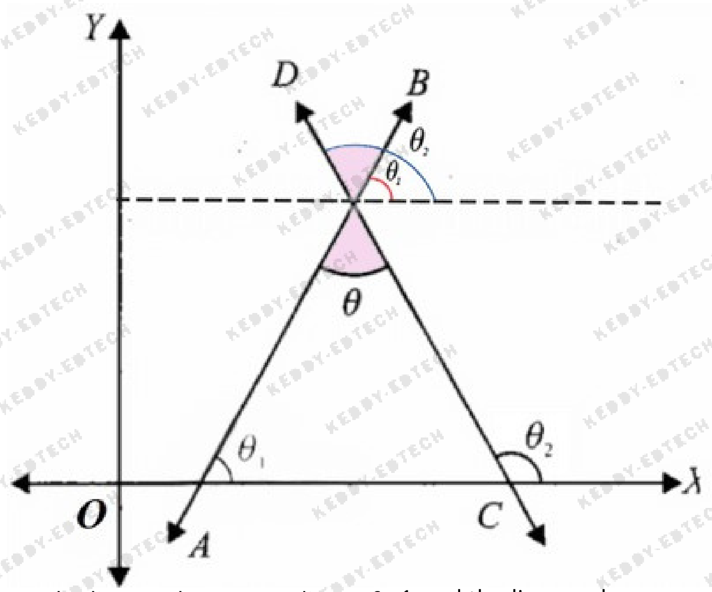
Angle between two lines

The angle, θ, between two lines having gradients $m1$ and $m2$ is given by

$$tan(\theta) = \pm \frac{m2 - m1}{1 + m1\,m2} \Rightarrow \vartheta = tan^{-1}\left(\pm \frac{m2 - m1}{1 + m1\,m2}\right)$$

- Note that you must take the positive value of the ratio if you are trying to find the acute angle between two lines.

Another way to look at this formula is by considering the formula to find θ using the gradient m



Let the line AB have a gradient of $m1$ and the line DC have a gradient of $m2$ such that

$$tan(\vartheta 1) = m1 \Rightarrow \vartheta 1 = tan^{-1}(m1)$$

$$tan(\vartheta 2) = m2 \Rightarrow \vartheta 2 = tan^{-1}(m2)$$

From the diagram above

$$\vartheta_2 - \vartheta_1 = \vartheta$$

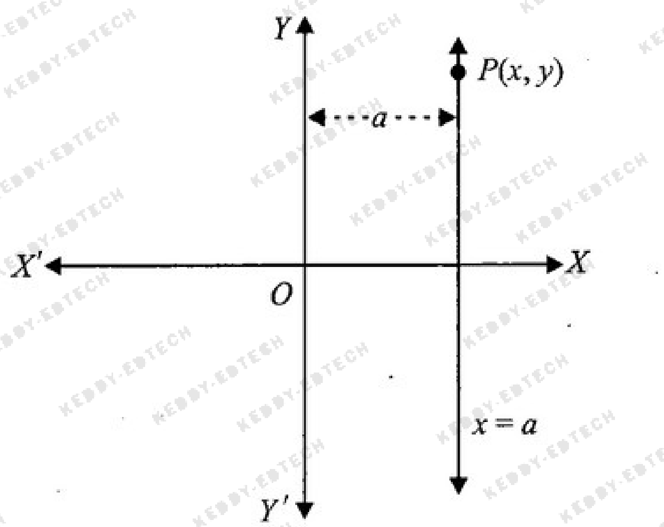
This gives us

$$tan^{-1}(m2) - tan^{-1}(m1) = \vartheta$$

1.3. Equations of lines

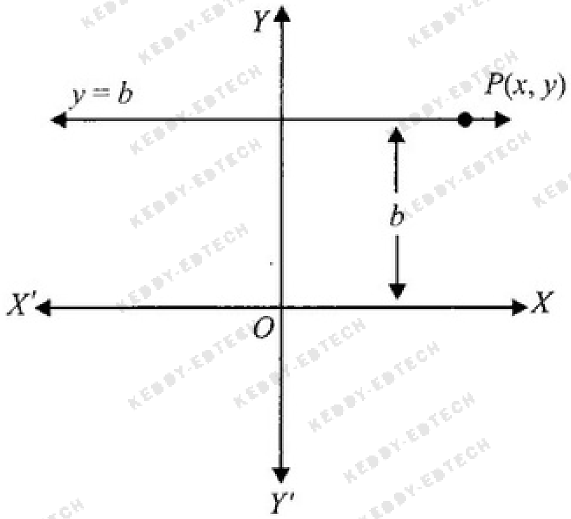
Equation of line parallel to y-axis

- The equation of a line parallel to the *y*-axis can be written in the form *x=a*.
  - All points on this line have the coordinate *P(a,y)*.



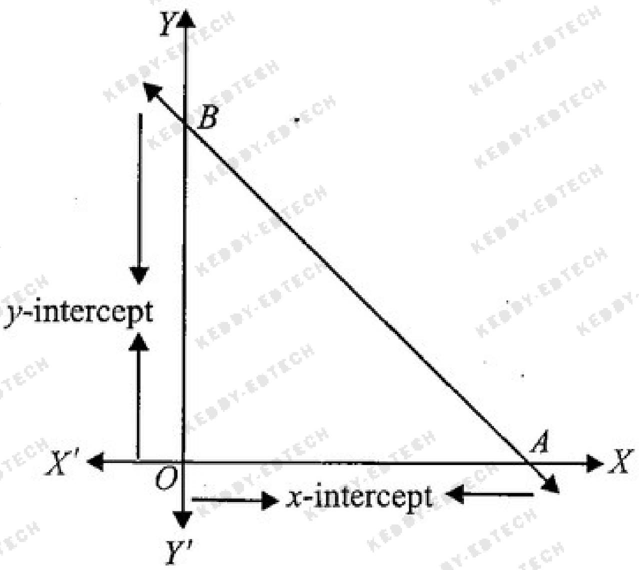
Equation of line parallel to x-axis

- The equation of a line parallel to the x-axis can be written in the form $y=b$.
  - All points on this line have the coordinate $P(x,b)$.



Intercepts

- An intercept is the point at which a line cuts the coordinate axes.



For this graph:

- The y-intercept is $(0,B)$
- The x-intercept is $(A,0)$

Equation of a straight line

The equation of a straight line is given by

$$y = mx + c$$

- m represents the gradient.
- c represents the y-intercept $(0,c)$. It can also be looked at in terms of a vertical translation.

Point-slope formula

The point-slope formula is another way to find the equation of a straight line.

- Consider a general point $P(x,y)$, which can be any point on the line, and a known point $Q(x_1, y_1)$ which the line passes through.

The gradient, m, can be found using the formula

$$m = \frac{y - y_1}{x - x_1}$$

From this, we find the point-slope formula by making (y - y_1) the subject of the formula

$$(y - y_1) = m(x - x_1)$$

Example

Find the equation of the straight line that passes through the points (-5, 3) and (-4, 1).

Answer

$$m = \frac{y_2 - y_1}{x_2 - x_1} = \frac{1 - 3}{-4 - (-5)}$$

$$m = -2$$

$$y = -2x + c$$

Now we can substitute any point in this equation to find the value of the y-intercept c

$$1 = -2(-4) + c$$

$$c = -7$$

So our final answer is

$$y = -2x - 7$$

1.4. Relation between lines

Relation between parallel lines

- Parallel lines have the same gradients. This is due to the angle between them being $0°$ degrees.

Consider two lines, $y=m_1x+c$ and $y=m_2x+c$. Given that these two lines are parallel, we can say:

$$m_1 = m_2$$

Relation between perpendicular lines

- Perpendicular lines create a $90°$ degree angle between them upon intersection. They are also called normal lines.

Consider two lines, $y=m_1x+c$ and $y=m_2x+c$. Given that these two lines are perpendicular, we can say:

$$m_1 \times m_2 = -1$$

Which means it is possible to find the gradient of one of the lines, given the gradient of the other line

$$m_2 = -\frac{1}{m_1} \text{ or } m_1 = -\frac{1}{m_2}$$

1.5. Perpendicular bisector

- A perpendicular bisector is a line that bisects another line perpendicularly, at a 90 degree angle, at its midpoint.

Example

Two points have coordinates *A*(5,7) and *B*(9,−1). Find the equation of the perpendicular bisector of the line *AB*.

Solution

The gradient, m_1, of *AB* can be calculated using

$$m = \frac{y_2 - y_1}{x_2 - x_1}$$

$$m = \frac{-1-7}{9-5}$$

$$m = -2$$

We can also find the gradient of the perpendicular, m_2, using

$$m_1 \times m_2 = -1$$

$$-2 \times m_2 = -1 \Rightarrow m_2 = \frac{1}{2}$$

So we have a line equation, for the perpendicular bisector, in the form

$$y = \frac{x}{2} + c$$

To find the value of the *y*-intercept, *c*, we can use the fact that the perpendicular bisector must intersect the midpoint of the line *AB*.

$$M = \left(\frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2}\right)$$

$$M = \left(\frac{5+9}{2}, \frac{7-1}{2}\right)$$

$$M = (7, 3)$$

Using these coordinates into our line equation, we get

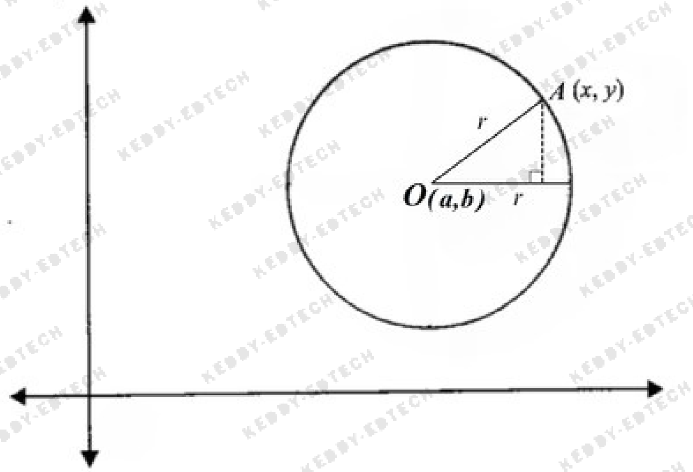
$$3 = \frac{7}{2} + c \Rightarrow c = -\frac{1}{2}$$

So the equation of the perpendicular bisector is

$$y = \frac{x}{2} - \frac{1}{2}$$

1.6. Equation of a circle

- The equation of a circle helps us describe a circle on the coordinate plane.



- Consider a circle with centre *O*(*a*,*b*) and radius *OA*, equal to *r*.
- A perpendicular line from *A*(*x*,*y*) can be drawn to depict a right angled triangle.
- This right angled triangle has:
  - *r* as its hypotenuse
  - (*x*−*a*) as its base
  - (*y*−*b*) as its height

Using the Pythagoras theorem

$$r^2 = (x-a)^2 + (y-b)^2$$

- Where the coordinates of the centre, *O*, of this circle are (*a*, *b*)

The equation of a circle can be represented in different ways, and it is important to correctly identify the coordinates of the centre from them.

Equation of circle	Coordinates of centre *O*
$r^2 = (x-a)^2 + (y-b)^2$	(a, b)
$r^2 = (x+a)^2 + (y+b)^2$	$(-a, -b)$
$r^2 = (x+a)^2 + (y-b)^2$	$(-a, b)$
$r^2 = (x-a)^2 + (y+b)^2$	$(a, -b)$

- A point, (*x*,*y*), lies on the circumference of the circle if it satisfies the equation of the circle when inputted.
  - Which means inputting points *x* and *y* into the equation of the circle will give you the value of r^2 if it lies on the circumference.

General equation of a circle

The expanded form of the equation of the circle is also known as the general equation of the circle.

$$0 = x^2 - 2ax + y^2 - 2by + c$$

Where $c = a^2 + b^2 - r^2$

Centre	(a,b)
Radius	$r = \sqrt{a^2 + b^2 - c}$

If the general form is given as

$$0 = x^2 + 2ax + y^2 + 2by + c$$

Centre	$(-a,-b)$
Radius	$r = \sqrt{a^2+b^2-c}$

- Another way to find the centre and radius is by completing the square, shown in the example 2 below.

Example 1

Points $A(-3,4)$ and $B(3,-4)$ lie on the circle, and the line AB is the diameter of that circle. Find the equation of the circle.

Answer

Since the diameter is the line AB, its midpoint must be the coordinates of the centre O of the circle.

$$M = \left(\frac{x_1+x_2}{2}, \frac{y_1+y_2}{2}\right)$$

$$O = \left(\frac{-3+3}{2}, \frac{4-4}{2}\right)$$

$O = (0, 0)$ [This means the centre lies on the origin]

As the points A and B lie on the circle, they must also satisfy the equation of the circle when inputted. Using this fact, we can input one of their coordinates to find the radius. Here we use the coordinates of A.

$$r^2 = (x-a)^2+(y-b)^2$$

$$r^2 = (-3-0)^2 + (4-0)^2$$

$$r^2 = 25 \Rightarrow r = 5$$

Thus, the equation of the circle can be written as

$$x^2 + y^2 = 5^2$$

Example 2

Find the center and the radius of the circle with the equation $x^2+y^2-10x-8y-40=0$.

Answer

Given the equation of the circle, we can complete the square for x and y to get

$$(x-5)^2-25+(y-4)^2-16-40=0$$

$$(x-5)^2+(y-4)^2=81$$

$$(x-5)^2+(y-4)^2=9^2$$

The centre of the circle is $(5, 4)$ and the radius is 9.

1.7. Translation applied to a Circle Equation

- A translation applied to a circle equation moves the coordinates of the centre and all the points on the circumference.
  - It does not change the radius, r.

Consider a circle equation

$$r^2 = (x-a)^2+(y-b)^2$$

With coordinates of the centre as (a,b). To apply a translation of $(c)\ d$, we can add this to the coordinates of the centre.

$$(a+c, b+d)$$

So the equation of the transformed graph is

$$r^2 = (x-[a+c])^2+(y-[b+d])^2$$

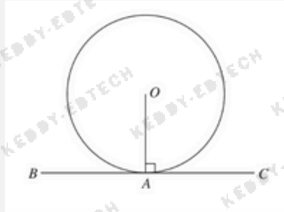
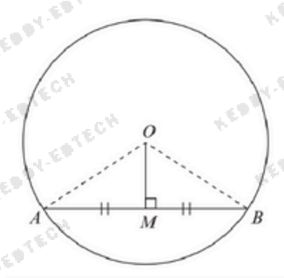
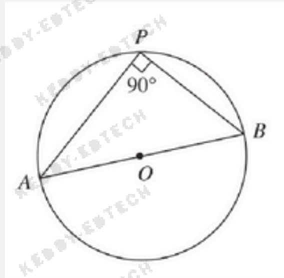
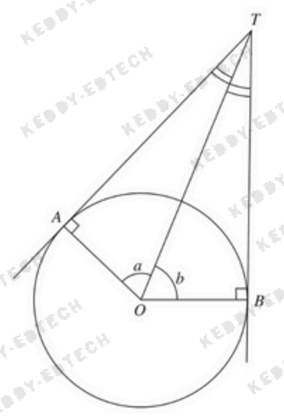
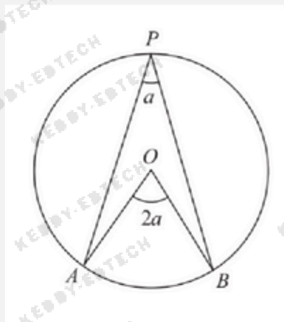
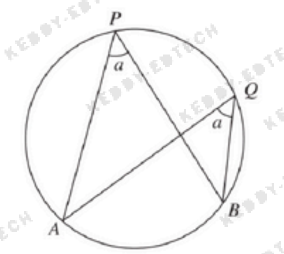
1.8. Intersection of lines and circles

- when an equation in the form of $ax^2+bx+c=0$ is created from the equation of the line and the circle, the value of b^2-4ac decides the number of intersection points
- if the value of b^2-4ac is
  - > 0 then two distinct points of intersection are formed
  - $= 0$ then a single point of intersection (line is a tangent to the circle) is formed
  - < 0 then there are no points of intersection

1. Circular measure

1.1. Circle theorems

These theorems must be known to solve circular measure questions.

Circle	Theorem
	Radius, *OA*, forms a 90° degree, or $\frac{\pi}{2}$ radian angle, at the point of contact with a tangent.
	The perpendicular bisector of a chord, *AB*, of a circle passes through the centre of the circle, *O*.
	The angle in a semi circle is always equal to 90° degrees or $\frac{\pi}{2}$ radians.
	Tangents which meet at the same point are equal in length, *TA*=*TB*, and angle *a*=angle *b*.
	The angle at the centre is twice the angle at the circumference.
	Angles in the same segment are equal.

1.2. Introduction to radians

- The radian is the angle formed when the arc length is the same as the length of the radius.
  - It is also denoted as a "rad".
- Consider a circle with its radius, r, equal to 1
  - An arc length of 2π would have an angle of $360°$ degrees, due to the circumference being $2\pi r$.
  - Half of its circumference, or an arc length of π, would have an angle of $180°$ degrees.

Converting between units

- Using the result, $180° = \pi$, we get these useful ratios
  - 1 radian = $\frac{180}{\pi}$ degrees
  - $1°$ degree = $\frac{\pi}{180°}$ radians

These ratios can help as convert from degrees to radians, or vice versa, as shown in the table below.

Converting from	Formula used
Degree to Radian	$\theta \times \frac{\pi}{180}$
Radian to Degree	radian $\times \frac{180°}{\pi}$

Example 1

Convert 360 degrees to radians.

Answer

$$360 \times \frac{\pi}{180°} = 2\pi \text{ radians}$$

Example 2

Convert $\frac{\pi}{2}$ radians to degrees.

Answer

$$\frac{\pi}{2} \times \frac{180°}{\pi} = 90 \text{ degrees}$$

1.3. Finding sector area and length

The formulae for finding sector area and sector length can also be used in terms of radians.

Finding sector area

We can find the sector area for a circle using:

Formula in degrees	Formula in radians
Area = $\frac{\theta}{360°} \times \pi r^2$	Area = $\frac{1}{2}\theta r^2$ [Where θ is in radians]

The formula in terms of radians can be derived easily using the fact $360° = 2\pi$

$$\frac{\theta}{360} \times \pi r^2 \Rightarrow \frac{\theta}{2\pi} \times \pi r^2$$

This simplifies to the formula in radians

$$\text{Area} = \frac{1}{2}\theta r^2 \text{ [Where } \theta \text{ is in radians]}$$

Finding sector length

We can find the sector length for a circle using:

Formula in degrees	Formula in radians
Length = $\frac{\theta}{360°} \times 2\pi r$	Length = θr [Where θ is in radians]

The formula in terms of radians can be derived easily using the fact $360° = 2\pi$

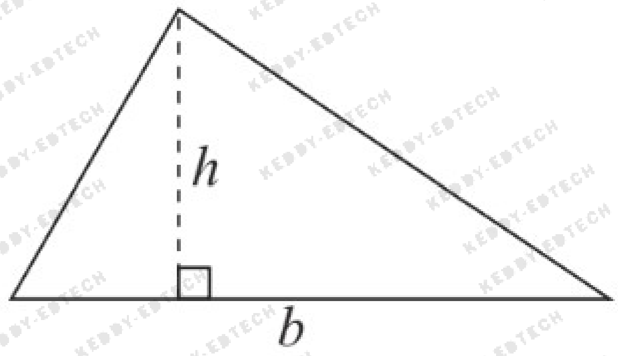
$$\frac{\theta}{360} \times 2\pi r \Rightarrow \frac{\theta}{2\pi} \times 2\pi r$$

This simplifies to the formula in radians

$$\text{Length} = \theta r \text{ [Where } \theta \text{ is in radians]}$$

1.4. Area of a triangle

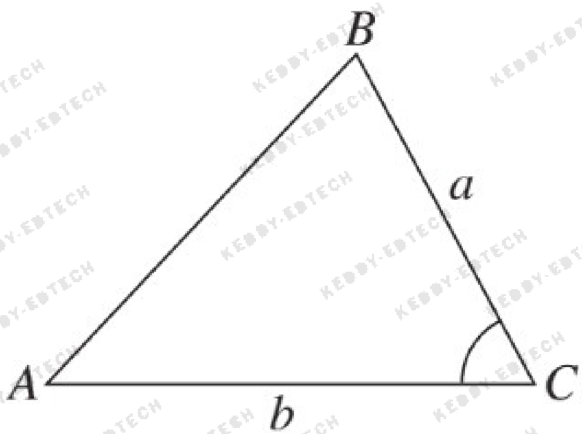
Let us take a triangle, where h represents the height and b represents the base.



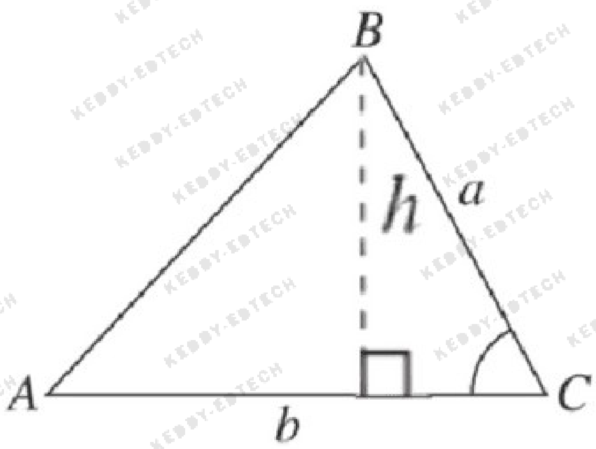
- The area of this triangle can be found using the formula $\frac{1}{2} \times base \times height$ or $\frac{1}{2} \times b \times h$.

Alternative formula for the area of a triangle

Consider another triangle, where you aren't given the base but have the values of 2 sides and 1 angle.



- To find the area of this triangle, we must first draw a perpendicular from B to the line AC



Utilising trigonometry, we observe that

$$\sin(C) = \frac{h}{a} \Rightarrow h = \sin(C) \times a$$

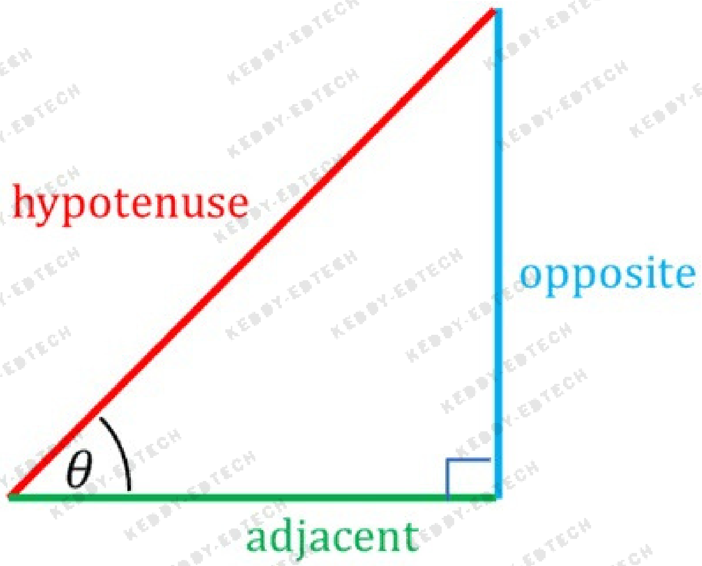
Taking the base as *b* and height as *sin*(*C*)×*a*, we can substitute it into the formula to find the area of this triangle as

$$\frac{1}{2} \times a \times b \times \sin(C)$$

- Note that this formula requires you to use an angle *C*, such that it is not facing the side *a* or side *b* as shown in the diagram above.

1. Trigonometry

1.1. Trigonometry ratios in right angle triangles (for θ between 0˚ and 90˚)



$$\sin(\theta) = \frac{\text{opposite}}{\text{hypotenuse}}$$

$$\cos(\theta) = \frac{\text{adjacent}}{\text{hypotenuse}}$$

$$\tan(\theta) = \frac{\text{opposite}}{\text{adjacent}}$$

- A mnemonic to remember the ratios is “SOH CAH TOA”.

Example

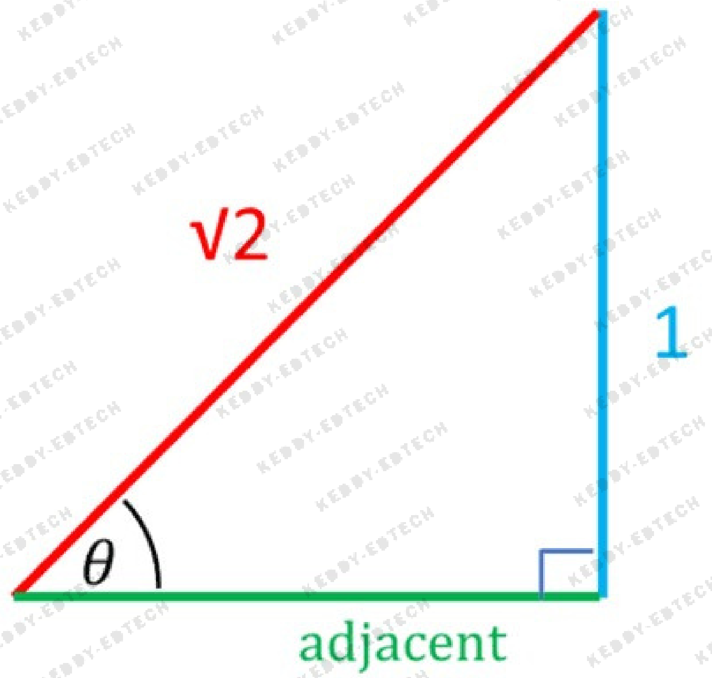
Find the ratio of $\cos(\theta)$, when $\sin(\theta) = \frac{1}{\sqrt{2}}$

Answer

Method 1

- Upon comparing $\sin(\theta)=\frac{1}{\sqrt{2}}$ and $\sin(\theta)=\frac{\text{opposite}}{\text{hypotenuse}}$, we note that the length of the opposite side is 1 and the length of the hypotenuse is $\sqrt{2}$.

We can now use the Pythagoras theorem to find the adjacent side of the triangle:



$$(\text{adjacent})^2+(1)^2=(\sqrt{2})^2 = 1$$

$$\cos(\theta)= \frac{\text{adjacent}}{\text{hypotenuse}} = \frac{1}{\sqrt{2}}$$

Method 2

Using the fact that

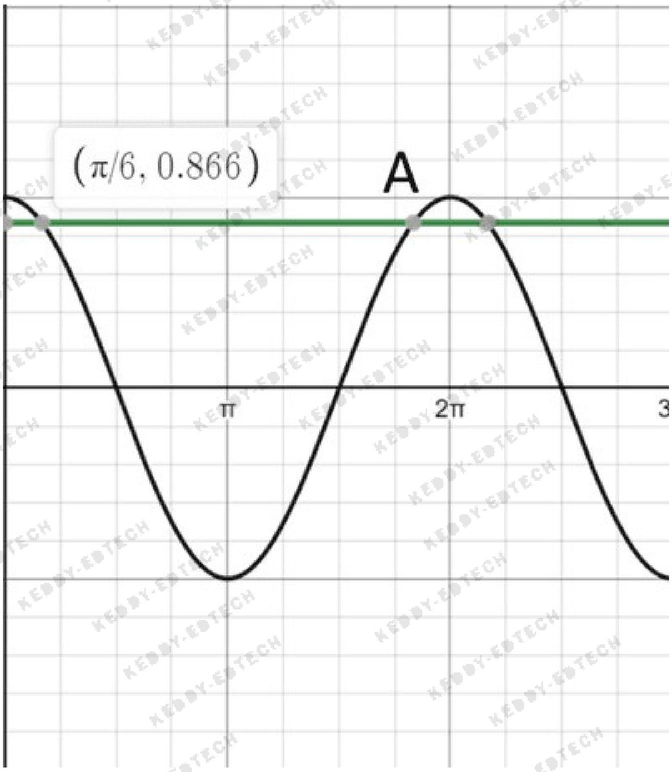
$$\sin(\theta)= \frac{1}{\sqrt{2}} \Rightarrow \theta=\sin^{-1}\left(\frac{1}{\sqrt{2}}\right) = 45°$$

$$\cos(45°)=\frac{1}{\sqrt{2}} \Rightarrow \cos(\theta) = \frac{1}{\sqrt{2}}$$

1.2. Find the other angles from the principle angle

- in case of $\cos(x)=\frac{\sqrt{3}}{2}$, we know using a calculator or by memory that $x=30°$ or $x=\frac{\pi}{6}$, but this is only the principle angle and there are many more solutions to this.

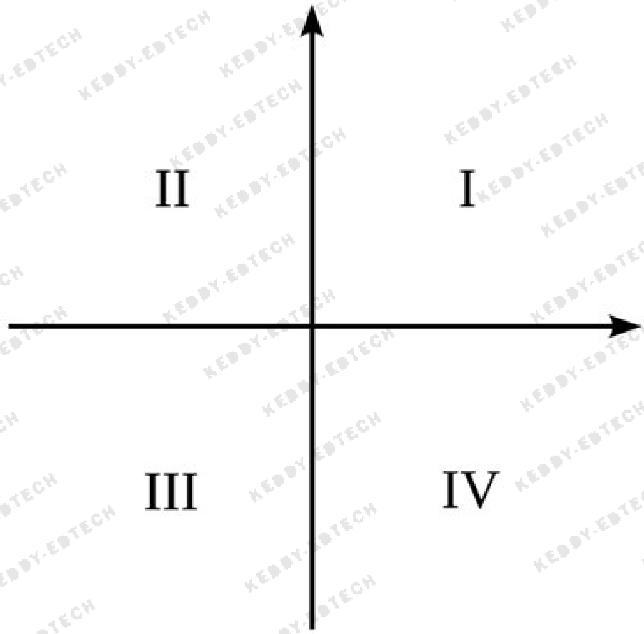
- to find the other values, draw a graph and draw a horizontal line for the value given to find the angles at which the trigonometric function give the value stated. in the previous example, the graph would look like this



- as you can see there are also other points on the graph which intersect with the horizontal line, these are also additional solutions
- in the question, a range might be given and you would have to find all the solutions in that given range.
- in this case, to find the other point *A* we can do $2\pi-\frac{\pi}{6}$, which equals to $\frac{11\pi}{6}$
- here we are using the property of trigonometry functions that they are symmetrical, and thus such calculations can be used to find the angles

1.3. Quadrants

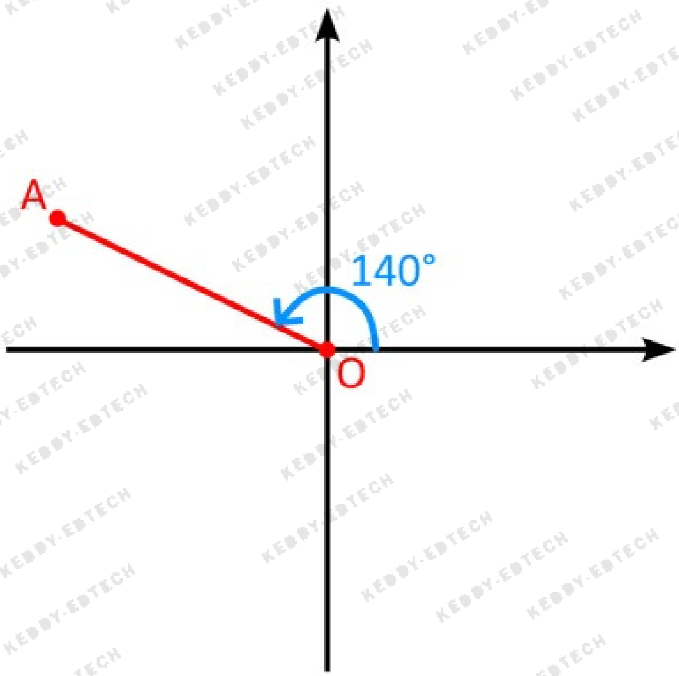
- The Cartesian plane is divided into 4 quadrants \n

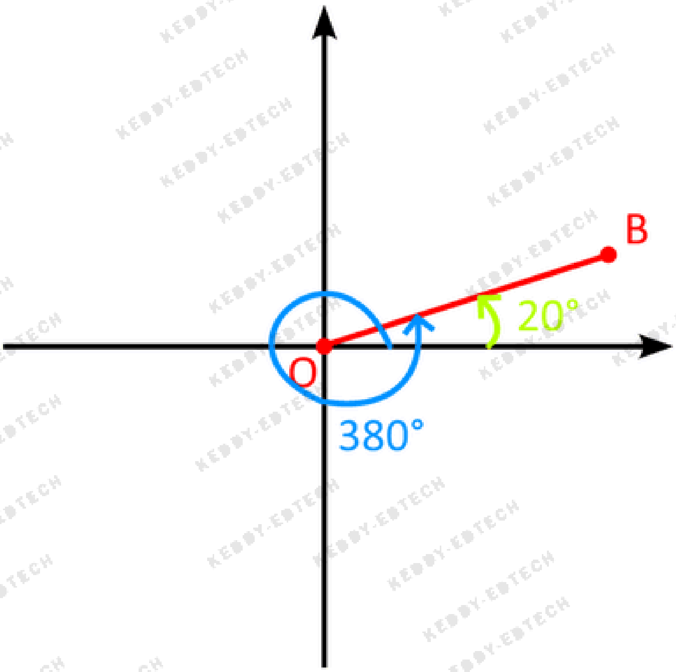


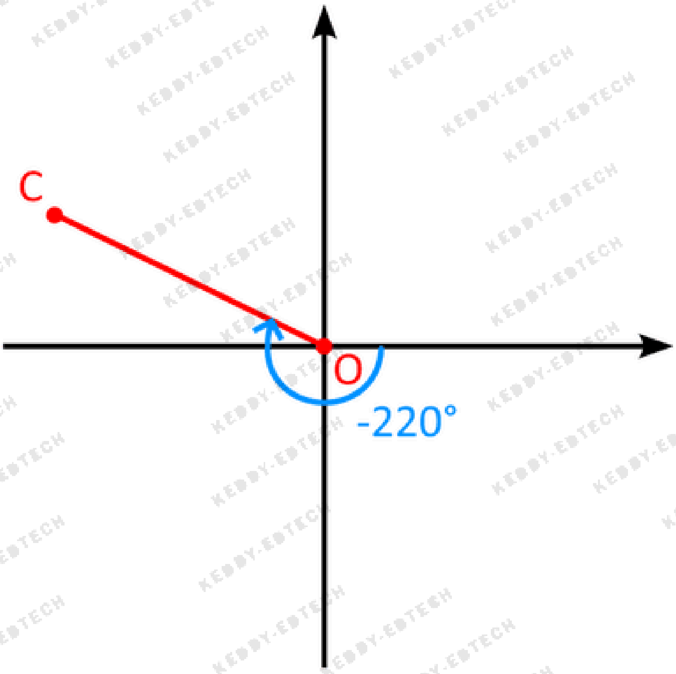
- an anticlockwise rotation from the x-axis of the first quadrant shows an increase in angle, but clockwise rotation shows a decrease in angle

Example

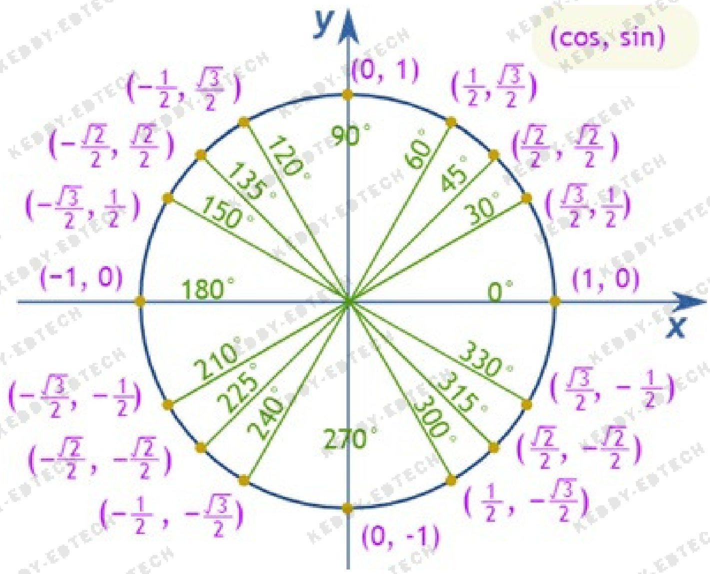
draw a line on a graph OA which is rotated 140°, a line OB rotated 380°, and a line OC rotated -220° Answer:





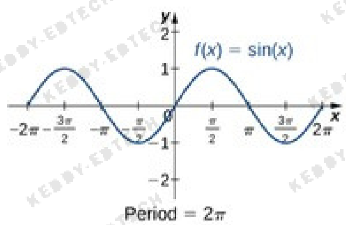


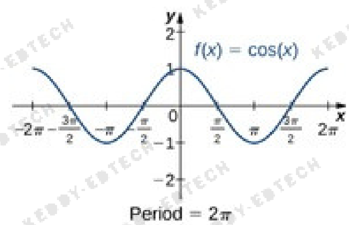
1.4. Quadrant - continued

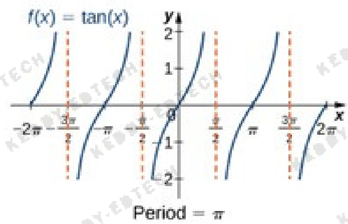


- First quadrant
  - $\cos(\theta) = +ve$
  - $\sin(\theta) = +ve$
  - $\tan(\theta) = +ve$
- Second quadrant
  - $\cos(\theta) = -ve$
  - $\sin(\theta) = +ve$
  - $\tan(\theta) = -ve$
- Third quadrant
  - $\cos(\theta) = -ve$
  - $\sin(\theta) = -ve$
  - $\tan(\theta) = +ve$
- Fourth quadrant
  - $\cos(\theta) = +ve$
  - $\sin(\theta) = -ve$
  - $\tan(\theta) = -ve$

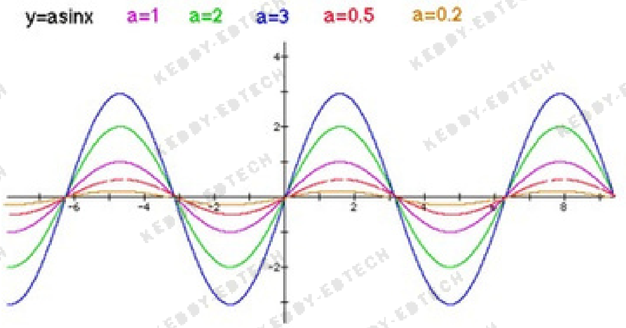
1.5. Graphs of Trigonometric functions



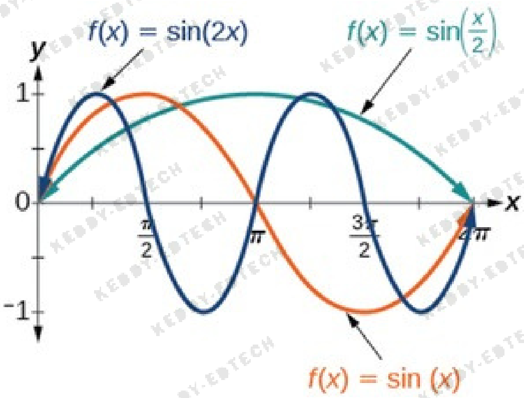




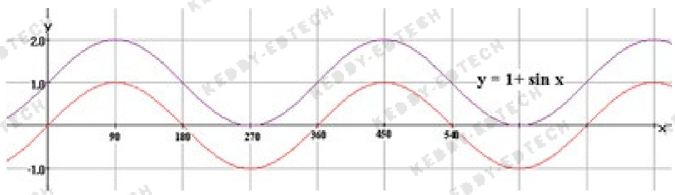
- period is the length of one cycle of the graph
- amplitude is the distance between the maximum or minimum points from the principal axis
- graph of $y=a\sin(x)$
  - The graph is stretched along the y-axis with a factor of a
  - The range of the graph changes to $a<y<-a$



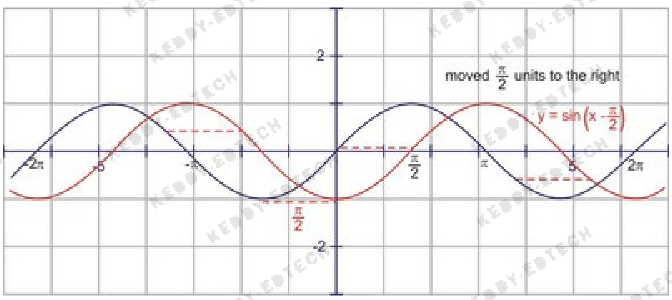
- graph of $y=\sin(ax)$
  - the period of the graph is $2\pi\ a$
  - the graph is stretched along the x-axis by a factor of $\frac{1}{a}$



- graph of $y=\sin(x)+a$
  - the graph is shifted along the y-axis by the value of a
  - translation of the graph by [0, a]
  - The range of the graph changes to $-1+a<y<1+a$



- graph of $y=\sin(x+a)$
  - the graph is shifted along the x-axis by the value of $-a$
  - translation of the graph by [$-a$, 0]



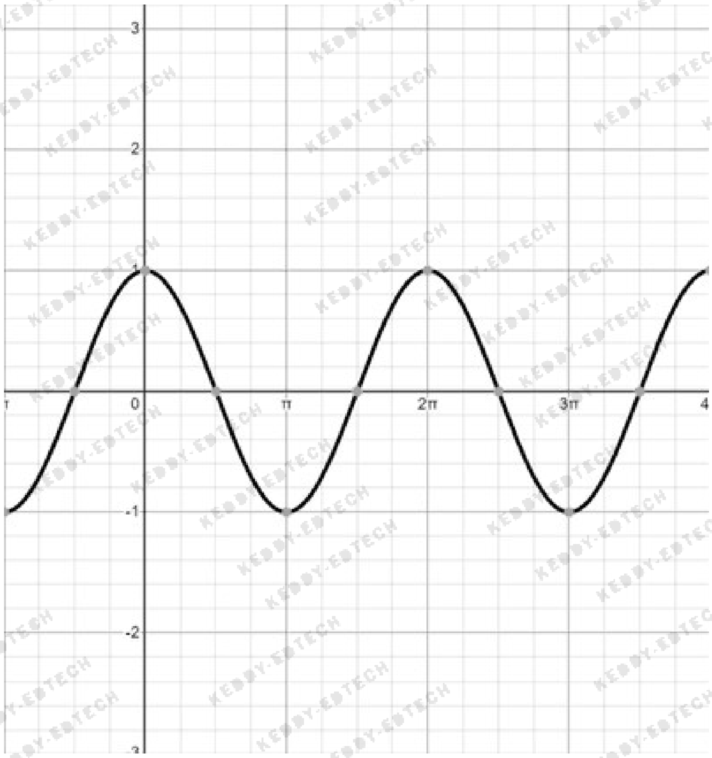
- When working with multiple transformations, go with this order:
  - firstly, work with the transformation inside the brackets of the trigonometric function
  - secondly, apply the transformation multiplied/divided to the trigonometric function
  - lastly, apply the transformation which is being added/subtracted to the trigonometric function.

Example

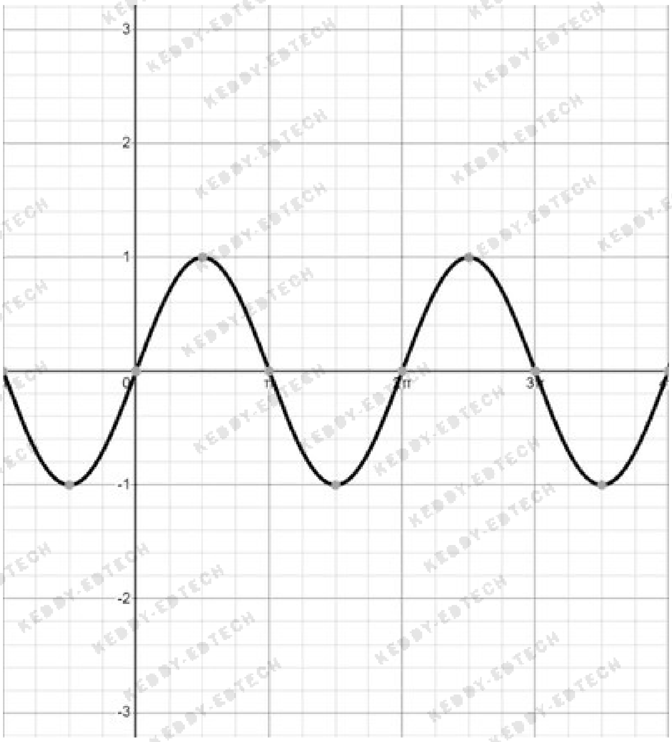
Draw the graph of $y = 3\cos(\frac{1}{2}x - \pi) - 1$

Answer

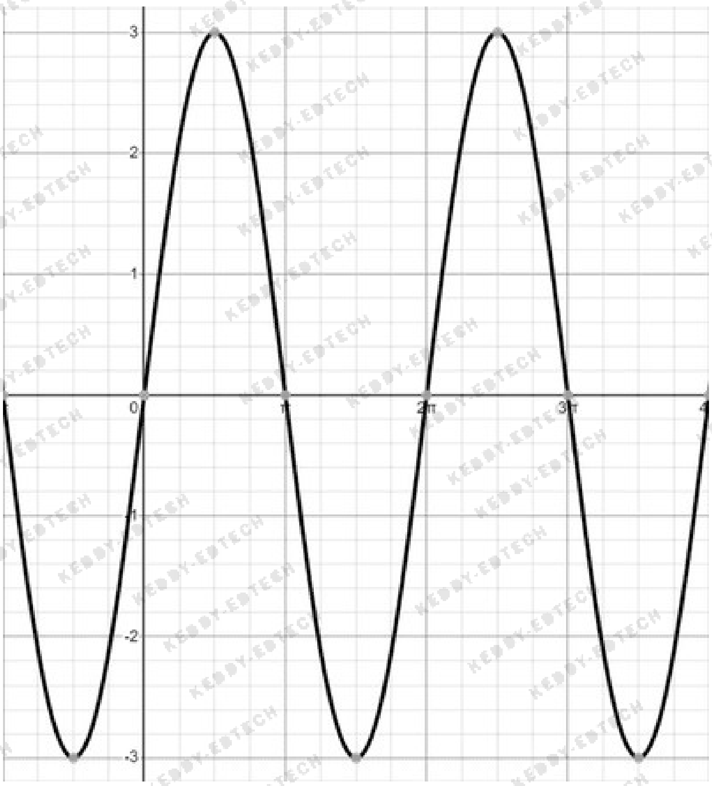
- draw the graph of $y = \cos(x)$



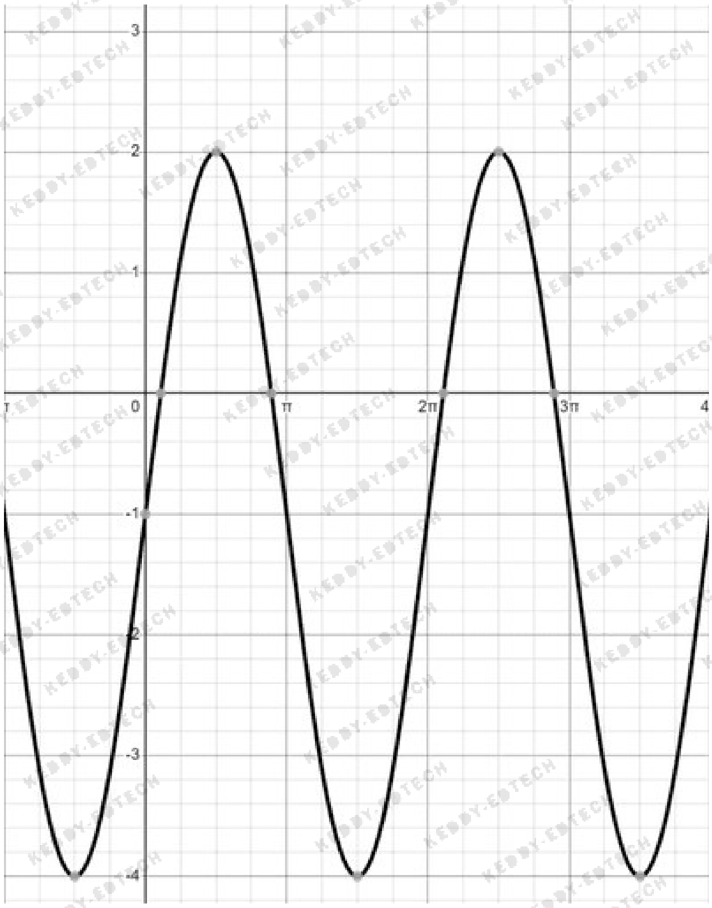
- apply the transformation inside the brackets $(x - 90)$, which translates the graph by $(90, 0)$



- apply the transformation which is being multiplied/divided to the trig function, in this case 3, which should stretch the graph by a factor of 3 along the y-axis



- finally, apply the transformation that is being added/subtracted to the expression, in this case it is -1, which would translate the graph by $(0, -1)$



1.6. Trigonometry identities

$$\tan(\theta) = \frac{\sin(\theta)}{\cos(\theta)}$$

$$\sin^2(\theta) + \cos^2(\theta) = 1$$

Example

Prove the identity $\frac{\cos 2(x) - \sin 2(x)}{\cos(x)} + \frac{1}{\cos(x)} \equiv 2\cos(x)$

Answer

$\sin 2(x)$ can be written as $1 - \cos 2(x)$

$$\frac{\cos 2(x) - (1 - \cos 2(x))}{\cos(x)} + \frac{1}{\cos(x)}$$

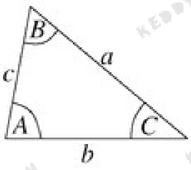
$$\frac{2\cos 2(x) - 1}{\cos(x)} + \frac{1}{\cos(x)}$$

$$\frac{2\cos 2(x) - 1 + 1}{\cos(x)}$$

$$\frac{2\cos 2(x)}{\cos(x)} \Rightarrow 2\cos(x)$$

1.7. Trigonometry with non right-angled triangles

- in these cases, you can either use the sine rule or the cosine rule



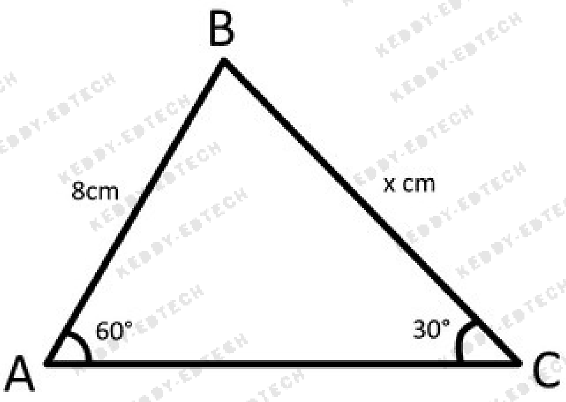
Sine Rule

$\frac{a}{\sin(A)} = \frac{b}{\sin(B)} = \frac{c}{\sin(C)}$ (for finding sides) or $\frac{\sin(A)}{a} = \frac{\sin(B)}{b} = \frac{\sin(C)}{c}$ (for finding angles)

Cosine Rule

$a^2 = b^2 + c^2 - 2bc\cos(A)$ (for finding sides) or $\cos(A) = \frac{b^2 + c^2 - a^2}{2bc}$ (for finding angles)

Example



Find the value of x

Answer

in this question, we are given two angles and one of the sides, where we have to find the length of the other side.

So we know we should use the sine rule $\frac{a}{\sin(A)} = \frac{b}{\sin(B)} = \frac{c}{\sin(C)}$

we can write this as $\frac{8}{\sin(30)} = \frac{x}{\sin(60)}$

cross multiplying it would give us

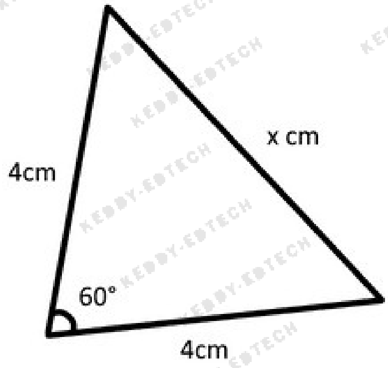
$8\sin(60) = x\sin(30)$

$8 \times \frac{\sqrt{3}}{2} = \frac{1}{2}x$

$8\sqrt{3} = x$

$\therefore x = 8\sqrt{3}$

Example



Find the value of x

Answer:

in this question, we are given two side lengths and the angle between them

in this case, we shall use the cosine rule

$a^2 = b^2 + c^2 - (2bc \times \cos(A))$

we can write this as $x^2 = 4^2 + 4^2 - (2 \times 4 \times 4 \times \cos(60))$

$x^2 = 16 + 16 - (16)$

$x^2 = 16$

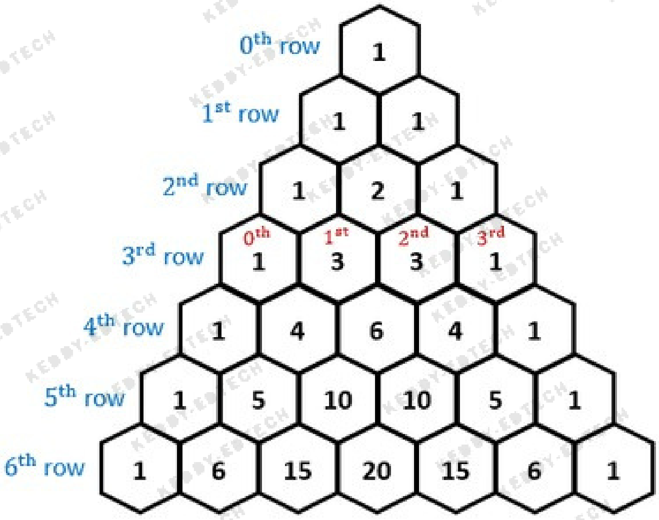
$\therefore x = 4$

1. Series

1.1. Introduction to nCr notation

- Combinatorics helps us find the *rth* element in the *nth* row of pascal's triangle, which is a triangle that is constructed by summing the adjacent elements in the previous row.

Pascal's triangle



- As an example, from the definition, $3C1$ will give us a value of 3.
- The triangle is also symmetrical at the centre

Factorial notation

The factorial notation is used in the formula for combinations.

$$n! = n \times (n-1) \times (n-2) \times \ldots \times 2 \times 1$$

For example

$$5! = 5 \times 4 \times 3 \times 2 \times 1 = 120$$

It should also be known that

$$0! = 1$$

Combinations

The notation you should be familiar for combinations is

$$^nC_r = \binom{n}{r} = \frac{n!}{r!(n-r)!}$$

Using this, we obtain some useful results

$$^nC_0 = {^nC_n} = 1$$

$$^nC_1 = {^nC_{n-1}} = n$$

$$^nC_r = {^nC_{n-r}}$$

- Reinstating, $\binom{n}{r}$ gives us the *rth* elementin the*nth* row of pascal's triangle.
- Most calculators have a pre-defined function, or button, that allows you to use the *nCr* formula directly.

1.2. Binomial expansion

- The binomial expansion helps us expand the sum of two terms raised to an exponent.
  - It makes use of pascal's triangle, as shown below, to obtain the binomial coefficient.

$$1 \rightarrow (x+y)^0 = 1$$
$$1 \quad 1 \rightarrow (x+y)^1 = x + y$$
$$1 \quad 2 \quad 1 \rightarrow (x+y)^2 = x^2 + 2xy + y^2$$
$$1 \quad 3 \quad 3 \quad 1 \rightarrow (x+y)^3 = x^3 + 3x^2y + 3xy^2 + y^3$$
$$1 \quad 4 \quad 6 \quad 4 \quad 1 \rightarrow (x+y)^4 = x^4 + 4x^3y + 6x^2y^2 + 4xy^3 + y^4$$
$$1 \quad 5 \quad 10 \quad 10 \quad 5 \quad 1 \rightarrow (x+y)^5 = x^5 + 5x^4y + 10x^3y^2 + 10x^2y^3 + 5xy^4 + y^5$$

The expansion is given in MF19 for $n > 0$, and can be generalised to

$$(a+b)^n = \binom{n}{0}a^n + \binom{n}{1}a^{n-1}b + \binom{n}{2}a^{n-2}b^2 + \ldots + \binom{n}{n-1}ab^{n-1} + \binom{n}{n}b^n$$

- Observations:
  - The exponents for *a* and *b* sum up to the value of *n*
  - The $\binom{n}{r}$ notation can be thought of as choosing *r* number of terms to be the second term, *b*.

We can find the rth term of a binomial expansion using

$$T_{r+1} = {^nC_r} \times a^{n-r} \times b^r$$

Binomial expansion with 3 terms

- We can also use binomial expansion to expand a trinomial, or the sum of three terms raised to an exponent.

Consider the expansion of $(a+b)^n$

$$(a+b)^n = \binom{n}{0}a^n + \binom{n}{1}a^{n-1}b + \ldots + \binom{n}{n-1}ab^{n-1} + \binom{n}{n}b^n$$

We can write the trinomial as

$$(x+y+z)^n \Rightarrow (a+[y+z])^n$$

Now we can substitute $b = y + z$ in the expansion of $(a+b)^n$

$$(a+b)^n = \binom{n}{0}x^n + \binom{n}{1}x^{n-1}(y+z) + \ldots + \binom{n}{n-1}x(y+z)^{n-1} + \binom{n}{n}(y+z)^n$$

Example

Find the expansion of $(3x+2)^4$.

Answer

$$(3x+2)^4 = (3x)^4 + \binom{4}{1}(3x)^{4-1}(2) + \binom{4}{2}(3x)^{4-2}(2)^2 + \binom{4}{3}(3x)^{4-3}(2)^3 + \binom{4}{4}(2)^4$$

$$(3x+2)^4 = 81x^4 + (4)(3x)^3(2) + (6)(3x)^2(4) + (4)(3x)(8) + 16$$

$$(3x+2)^4 = 81x^4 + 216x^3 + 216x^2 + 96x + 16$$

1.3. Arithmetic Progression

- An arithmetic progression (AP) is a series of numbers that share a common difference between consecutive terms
  - An example of an arithmetic progression is: 2,4,6,8,…. Each consecutive term in this series has a common difference of 2.
  - The first term is denoted by a and the common difference is denoted by d

$$1st \text{ term} = u_1 = a$$

$$2nd \text{ term} = u_2 = a + d$$

$$3rd \text{ term} = u_3 = a + 2d$$

$$\dots$$

$$(n-1)^{th} \text{ term} = u_{n-1} = a + (n-2)d$$

$$n^{th} \text{ term} = u_n = a + (n-1)d$$

$$(n+1)^{th} \text{ term} = u_{n+1} = a + (n+1)d$$

- The formula for the n^{th} term is also given in the MF19 formula sheet.

Finding the common difference

As mentioned previously, the common ratio must be the same between consecutive terms.

This can be written as

$$d = u_2 - u_1 = u_3 - u_2 = \dots = u_n - u_{n-1} = u_{n+1} - u_n$$

We can re-write this as

$$d = a + d - a = a + 2d - (a + d)$$

$$\dots$$

$$d = a + (n-1)d - [a + (n-2)d] = a + (n+1)d - [a + (n-1)d]$$

- Note that the difference of two consecutive terms results in d being the only term left.

Sum of Arithmetic Progression

- The sum of a geometric progression is the sum of terms in a given sequence.
  - S_n denotes the sum till the nth term.

It is given in MF19, and is denoted by

$$S_n = \frac{n}{2}[2a + (n-1)d]$$

An alternate formula also exists

$$S_n = \frac{n}{2}(a + l)$$

Where l denotes the last term in the series

- The use of both formulae depends on the context, and information about the variables.

Example 1

Find the 120th term of the arithmetic progression: 2,4,6,8,10...

Answer

- From the question
  - The first term is equal to a, which is 2 in this question.
  - The common difference, d, is equal to the difference between consecutive terms, 4−2=2.

So to find the 120th term

$$u_{120} = 2 + (120 - 1)2 = 240$$

Example 2

Find the sum of the first 20 terms of the arithmetic progression, which has its first term equal to 7 and its 8th term being 28.

Answer

From the question, we can deduce that a=7 as it is the first term. Using the general formula for the nth term for an AP

$$u_8 = 28 = 7 + (8-1)d$$

$$d = \frac{28-7}{7} = 3$$

Thus, we can use the sum till the n^{th} to find the sum of the first 20 terms.

$$S_{20} = \frac{20}{2}[2(7) + (20-1)3] = 710$$

1.4. Geometric Progression

- A geometric progression (GP) is a series of numbers that share a common ratio between consecutive terms.
  - An example of a geometric sequence is: 3,9,27,81,…. Each term in this series is being multiplied by a common ratio of 3.
  - The first term is given by a, while the common ratio is denoted by r

$$1st \text{ term} = u_1 = a$$

$$2nd \text{ term} = u_2 = ar$$

$$3rd \text{ term} = u_3 = ar^2$$

$$\dots$$

$$(n-1)^{th} \text{ term} = u_{n-1} = ar^{n-2}$$

$$n^{th} \text{ term} = u_n = ar^{n-1}$$

$$(n+1)^{th} \text{ term} = u_{n+1} = ar^n$$

- The formula for the n^{th} term is also given in the MF19 formula sheet

Finding the common ratio

As mentioned previously, the common ratio must be the same between consecutive terms.

This can be written as

$$r = \frac{u_2}{u_1} = \frac{u_3}{u_2} = \dots = \frac{u_n}{u_{n-1}} = \frac{u_{n+1}}{u_n}$$

We can re-write this ratio as

$$r = \frac{ar}{a} = \frac{ar^2}{ar} = \dots = \frac{ar^{n-1}}{ar^{n-2}} = \frac{ar^n}{ar^{n-1}}$$

- Note that the denominator cancels all the terms of the numerator, and leaves r as the only term.

Sum of Geometric Progression

- The sum of a geometric progression is the sum of terms in a given sequence.
  - S_n denotes the sum till the nth term.

It is given in MF19, and is denoted by

$$S_n = \frac{a(1-r^n)}{1-r} \quad [r < 1]$$

Another formula exists for the case r > 1

$$S_n = \frac{a(r^n-1)}{r-1} \quad [r>1]$$

- The sum till infinity of a geometric progression, given in MF19, is denoted by

$$S_\infty = \frac{a}{1-r} \quad [-1<r<1]$$

- The inequality, $-1<r<1$, is also written as $|r| <1$ but implies the same.
  - For a geometric progression to converge, or reach a finite value when the sum is taken till infinity, it must satisfy this inequality.

Example 1

Find the $10th$ term of the geometric progression, with the first term 3 and a ratio of $\frac{3}{4}$.

Answer

- From the question:
  - $a=3$ and $r=\frac{3}{4}$, giving us $u_n = 3(\frac{3}{4})^{n-1}$

So to find the $10th$ term,

$$u_{10} = 3(\frac{3}{4})^{10-1} \Rightarrow 0.225$$

Example 2

The first 3 terms of a geometric progression are: $x+2$, $3x$, $\frac{9}{4}x^2$, where $x>0$, show that $9x^3-18x^2=0$ and find the value of r and x.

Answer

We know the common ratio, r, is the ratio of any two consecutive terms

$$r = \frac{3x}{x+2} = \frac{\frac{9}{4}x^2}{3x}$$

Now we can begin simplifying our ratio

$$\frac{9x^2}{x+2} = \frac{9}{4}x^2$$

$$4(9x^2) = 9x^2(x+2)$$

$$36x^2 = 9x^3 + 18x^2$$

$$\Rightarrow 9x^3 - 18x^2 = 0$$

We can now find the value of r and x

$$9x^3 = 18x^2$$

$$\frac{x^3}{x^2} = \frac{18}{9} \Rightarrow x = 2$$

Substituting the value of x back into the common ratio gives us:

$$r = \frac{3(2)}{2+2} = \frac{3}{2}$$

1.5. Sum of a series

As introduced previously, the sum of a series adds up each term in a sequence till the nth term.

Using the general definition of the sum, we obtain

$$S_{n-1} = u_1 + u_2 + \dots + u_{n-3} + u_{n-2}$$

$$S_n = u_1 + u_2 + \dots + u_{n-2} + u_{n-1}$$

$$S_{n+1} = u_1 + u_2 + \dots + u_{n-1} + u_n$$

Using these basic definitions we can derive two important results

$$S_n - S_{n-1} = u_{n-1}$$

$$S_{n+1} - S_n = u_n$$

- This can help you determine the common difference or ratio of a sequence

It is also important to note that:

$$S_1 = a \quad [a \text{ denotes the first term of the sequence here}]$$

Sum of a series given in quadratic form

- Many questions may give you the sum in terms of a quadratic

$$S_n = An^2 + Bn \quad \text{[Where A and B are constants]}$$

- A sum given in this form is always an arithmetic progression.

One of the methods to solve such questions is using the definitions and results that arise from the sum of a series.

- An alternate way is to compare the coefficients with the sum of an arithmetic progression

$$S_n = \frac{n}{2}[2a+(n-1)d] = An^2 + Bn$$

$$\frac{dn^2}{2} - \frac{dn}{2} + an = An^2 + Bn$$

Upon comparing coefficients, we get:

$$\frac{d}{2} = A \Rightarrow d = 2A$$

$$-\frac{d}{2} + a = B \Rightarrow a = B + A$$

- Where a is the first term of the arithmetic sequence and d is the common difference.

Finding sum in-between 2 terms

- Recall that the formula for the sum of a series gives you the sum till the *nth* term.

To find the sum in-between two terms, say a and b [where $b>a$], we can use the fact that S_b includes S_a to some extent.

- To take an example, the sum of numbers between the 20*th* and 100*th* term for a sequence is given by:

$$S_{100} = u_1 + u_2 + \ldots + u_{19} + u_{20} + \ldots + u_{99} + u_{100}$$

$$S_{19} = u_1 + u_2 + \ldots + u_{18} + u_{19}$$

We can clearly deduce that

$$S_{100} - S_{19} = u_{20} + u_{21} + \ldots + u_{99} + u_{100}$$

Another way to think about this is

$$S_{100} = S_{19} + u_{20} + u_{21} + \ldots + u_{99} + u_{100}$$

So subtracting S_{19} from S_{100} gives us the sum between the 20^{th} and 100*th* term.

1.6. Trigonometry in series

- Trigonometric functions can be included in series and sequences.
  - They make use of identities such as $\tan(\theta) = \frac{\sin(\theta)}{\cos(\theta)}$ and $\sin^2(x) + \cos^2(x) = 1$
  - These series can also exist for a specific domain, such as $0 \le x \le \pi$, but questions will usually specify the value of x or θ to use.

Trigonometric Arithmetic Progression

Example 1

Given an arithmetic sequence with first term $\sin^2(\theta)$, where $0<\theta<\frac{\pi}{2}$, and second term $\sin^2(\theta)\cos^2(\theta)$, find the common difference in terms of $\sin(\theta)$

Answer

- From the question
  - $a = \sin^2(\theta)$ and $a + d = \sin^2(\theta)\cos^2(\theta)$

Using the fact that $u_{n+1} - u_n = d$

$$\sin^2(\theta)\cos^2(\theta) - \sin^2(\theta) \Rightarrow \sin^2(\theta)(\cos^2(\theta) - 1)$$

After noticing that $\cos^2(\theta) - 1 = -\sin^2(\theta)$

$$\sin^2(\theta)(-\sin^2(\theta)) \Rightarrow d = -\sin^4(\theta)$$

Example 2

Given an arithmetic sequence with the *nth* term $u_n = \sin^2(\theta) + (n-1)(-\sin^4(\theta))$, find the sum of the first 16 terms when $\theta = \pi/3$.

Answer

Using $\theta = \frac{\pi}{3}$, we get

$$a = \sin^2\left(\frac{\pi}{3}\right) = \frac{3}{4}$$

$$d = -\sin^4\left(\frac{\pi}{3}\right) = -\frac{9}{16}$$

Now we can use the formula for the sum till the n^{th} term

$$S_n = \frac{n}{2}[2a + (n-1)d]$$

$$S_{16} = \frac{16}{2}\left[2\left(\frac{3}{4}\right) + (16-1)\left(-\frac{9}{16}\right)\right] = -\frac{111}{2}$$

Trigonometric Geometric Progression

Example

Given a geometric progression with first term $\sin^2(\theta)$, where $0 < \theta < 2\pi$, and second term $\sin^2(\theta)\cos^2(\theta)$, find the sum till infinity.

Answer

- From the question
  - We know that $a = sin2(\theta)$ and $ar = sin2(\theta)cos2(\vartheta)$

To find the common ratio r:

$$r = \frac{sin2(\theta)cos2(\vartheta)}{sin2(\theta)} = cos2(\vartheta)$$

We can now use the sum till infinity formula for a geometric progression

$$S_\infty = \frac{sin2(\theta)}{1 - cos2(\theta)}$$

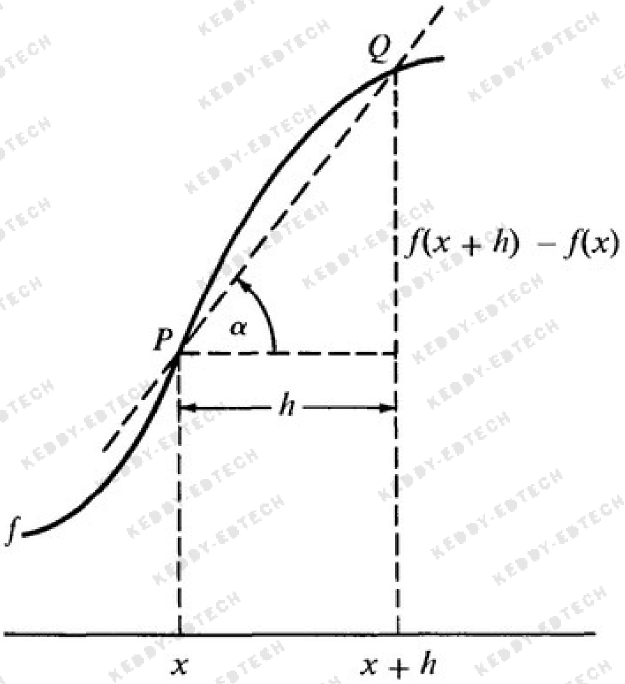
After noticing that $1 - cos2(\vartheta) = sin2(\theta)$

$$S_\infty = \frac{sin2(\theta)}{sin^2(\theta)} = 1$$

1. Differentiation

1.1. Introduction

- Differentiation helps us find the gradient of the tangent, a line which touches the curve at a single point.



Using the formula for finding the gradient, we get

$$m = \frac{f(x+h) - f(x)}{h}$$

Where *h* is a small number that is greater than 0.

- As *h* becomes smaller and approaches 0 (but never reaches it), we get the gradient of a tangent to a point on the curve.

From this, we can deduce

$$f'(x) = \tan(\theta) = m$$

- Where $f'(x)$ represents the first derivative, or the result obtained by differentiating $f(x)$ once.

Notation

It is denoted by many notations:

$$\frac{dy}{dx} = f'(x) = y'$$

- Where $\frac{dy}{dx}$ implies a function y has been differentiated with respect to the variable *x*.
  - It can also be written as $\frac{d}{dx}y$, where *y* is the function being differentiated.

Basics of Differentiation

Power rule

$$\frac{d}{dx}x^n = n \times x^{n-1} \quad \text{[For any real number } n \neq 0]$$

Sum and Difference rule

$$\frac{d}{dx}f(x) \pm g(x) = \frac{d}{dx}f(x) \pm \frac{d}{dx}g(x)$$

Scalar multiple rule

$$\frac{d}{dx}a \times f(x) = a \times \frac{d}{dx}f(x) \quad \text{[Where } a \text{ is a constant number]}$$

Constant rule

$$\frac{d}{dx}a = 0 \quad \text{[Where } a \text{ is a constant number]}$$

1.2. Chain rule

- The chain rule helps us differentiate a composite function.

Consider *f*(*g*(*x*)). If we let *u*=*g*(*x*) and *y*=*f*(*u*), and then differentiate *y*, we get

$$\frac{dy}{dx} = \frac{dy}{du} \times \frac{du}{dx}$$

So we can define the derivative of $y = f(g(x))$

$$\frac{d}{dx}f(g(x)) = f'(g(x)) \times g'(x)$$

It may also be thought of like this

$$\underbrace{f'}_{\text{derivative of outside function}} \quad \underbrace{(g(x))}_{\text{inside function left alone}} \quad \underbrace{g'(x)}_{\text{times derivative of inside function}}$$

Applications of chain rule

- The chain rule can be applied to differentiate functions of the type $[f(x)]^n$

Using the chain rule and power rule, we can differentiate $y = [f(x)]^n$ to get

$$\frac{d}{dx}[f(x)]^n = n \times [f(x)]^{n-1} \times f'(x)$$

Another application is seen while differentiating $y = (ax+b)^n$

$$\frac{d}{dx}(ax+b)^n = n \times (ax+b)^{n-1} \times \frac{d}{dx}ax+b$$

$$\Rightarrow \frac{d}{dx}(ax+b)^n = n \times (ax+b)^{n-1} \times a$$

- This is valid for all real numbers $n \neq 0$

Example

Differentiate $y = (2x+5)^5$.

Answer

Using the chain rule

$$\frac{d}{dx}(2x+5)^5 = 5 \times (2x+5)^{5-1} \times \frac{d}{dx}2x+5$$

$$\Rightarrow \frac{d}{dx}(2x+5)^5 = 5 \times (2x+5)^4 \times 2$$

$$\Rightarrow \frac{d}{dx}(2x+5)^5 = 10 \times (2x+5)^4$$

1.3. Gradient function

- The first derivative is also called the gradient function. It gives us the gradient of the tangent to the curve at a specific point.

A general equation of a tangent can be found using

$$y = f'(x)x + c$$

To find the gradient of the tangent at the point (a, b), input x = a into the first derivative.

$$f'(a)$$

Using this, we can re-write the equation of the tangent to the curve at the point (a, b) as

$$y = f'(a)x + c$$

Equation of perpendicular

- To find the equation of a perpendicular to a point, we must first find the equation of the gradient of the tangent at that point on the curve.

We can consider the gradient of the tangent at (x,y) to be $f'(x)$

Using the fact that the gradients of a perpendicular and a tangent follow the relation:

$$m1 \times m2 = -1$$

We can re-write the gradient of the perpendicular as

$$f'(x) \times m2 = -1 \Rightarrow m2 = -\frac{1}{f'(x)}$$

Example 1

Find the gradient of the tangent to the curve $y = 5x^2 + 3x$ at the point $x=2$.

Answer

We first start by finding the gradient function, or the first derivative.

$$\frac{d}{dx}\, 5x^2 + 3x = 10x + 3$$

Now we can input $x = 2$ into $f'(x)$

$$f'(2) = 10(2)+3 = 23$$

So the gradient of the tangent to the curve at $x = 2$ is 23.

Example 2

A curve is described by the equation

$$f(x) = \frac{1}{\sqrt{x}} \text{ for } x > 0$$

The point A is on the curve such that the normal to the curve at A also passes through the origin. Find the coordinates of the point A

Answer

- With the given information, the point A can be considered the point of intersection of the curve $f(x)$ and the perpendicular to the curve at that point.

To find the gradient of the normal, we first find the general gradient of a tangent to the curve by differentiating the curve once.

$$f'(x) = \frac{d}{dx} x^{-\frac{1}{2}} = -\frac{1}{2x^{\frac{3}{2}}}$$

Now we can use the relationship between the gradients of a perpendicular and a tangent

$$-\frac{1}{2x^{\frac{3}{2}}} \times m2 = -1 \Rightarrow m2 = -\frac{1}{-\frac{1}{2x^{\frac{3}{2}}}}$$

So the gradient of the normal is

$$m2 = 2x^{\frac{3}{2}}$$

Using this gradient, and the y-intercept as 0 (due to the question stating that the normal passes through the origin), we get the equation of the normal as

$$y = mx + c \Rightarrow y = 2x^{\frac{3}{2}} \times x + 0$$

$$y = 2x^{\frac{5}{2}}$$

Equating $f(x)$ and the equation of the perpendicular will give us the x coordinate of A.

$$x^{-\frac{1}{2}} = 2x^{\frac{5}{2}} \Rightarrow x = 2^{-\frac{1}{3}}$$

Inputting this x-value into any one equation will give us the corresponding y-coordinate of A.

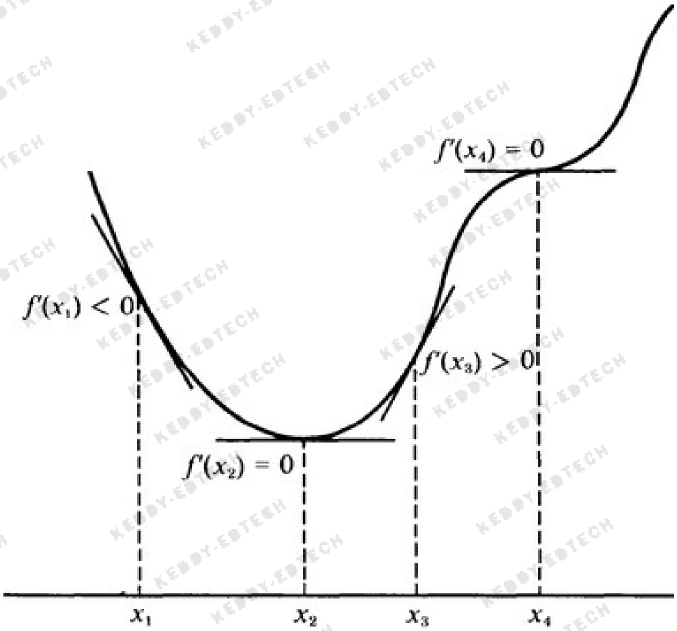
$$y = (2^{-\frac{1}{3}})^{-\frac{1}{2}} \Rightarrow y = 2^{\frac{1}{6}}$$

So the coordinates of A are

$$(2^{-\frac{1}{3}}, 2^{\frac{1}{6}})$$

1.4. Rate of change

- The rate of change of a function can be connected to the rate of change of its gradient.
  - A positive gradient means the function has an increasing *y*-value as the *x*-value increases. This is called an increasing function.
  - A negative gradient means the function has a decreasing *y*-value as the *x*-value increases. This is called a decreasing function.
  - A gradient of zero means the function is neither decreasing or increasing.

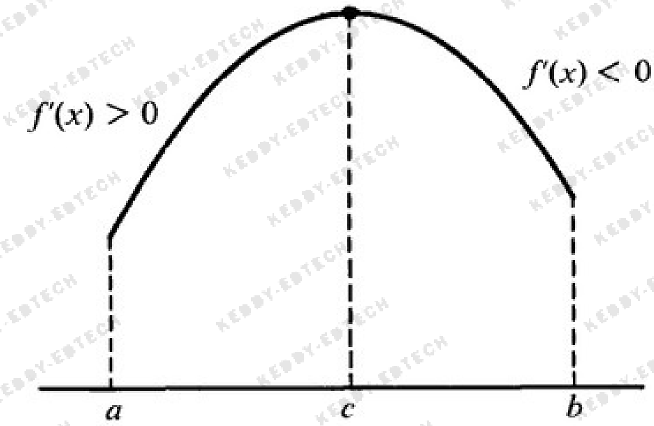


Using the graph above, we can re-write the rate of change as

Gradient function - $f'(x)$	Gradient	Function - $f(x)$
$f'(x) > 0$	Increasing	Increasing function
$f'(x) = 0$	Stationary	Stationary point
$f'(x) < 0$	Decreasing	Decreasing function

- It is also important to note that $f'(x) > 0$ does not always imply $f(x) > 0$, vice versa.

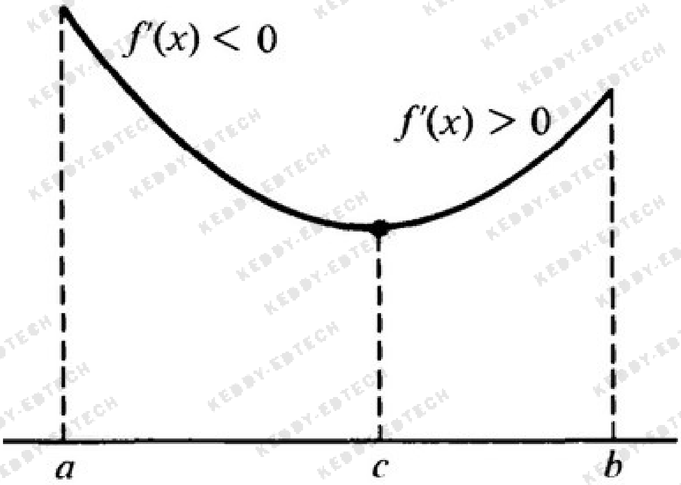
Maximum point



Given a maxima with *x* coordinate *c*:

Left of maxima	Right of maxima
Increasing gradient $f'(x) > 0$	Decreasing gradient $f'(x) < 0$
Function is increasing	Function is decreasing

Minimum point



Given a minima with *x* coordinate *c*:

Left of minima	Right of minima
Decreasing gradient $f'(x) < 0$	Increasing gradient $f'(x) > 0$
Function is decreasing	Function is increasing

Example 1

A function is defined by

$$f(x) = \frac{5}{1-3x} \text{ for } x \geq 1$$

Determine wether *f* is an increasing function, decreasing function, or neither.

Answer

First we find the first derivative, or the gradient function, of *f*(*x*) using the chain rule.

$$\frac{d}{dx}5(1-3x)^{-1} = -1\times5\times-3\times(1-3x)^{-2}$$

$$f'(x) = 15\times(1-3x)^{-2}$$

As:

- The 15 being multiplied is positive, it does not change the sign of the function.
- $(1-3x)^{-2}$ is always positive as it is squared, it does not change the sign of the function.

Using this logic, we can deduce that

$$f'(x) > 0$$

So $f(x)$ is an increasing function for all real values of x.

Example 2

Find the set of values of x for which the function $f(x) = \frac{(2x-3)^3}{6} - 4x$ is increasing.

Answer

We start by finding the first derivative, or gradient function, of $f(x)$ using the chain rule.

$$\frac{1}{6}\times\frac{d}{dx}(2x-3)^3 + \frac{d}{dx}-4x = \frac{1}{6}(3\times2\times(2x-3)^2) - 4$$

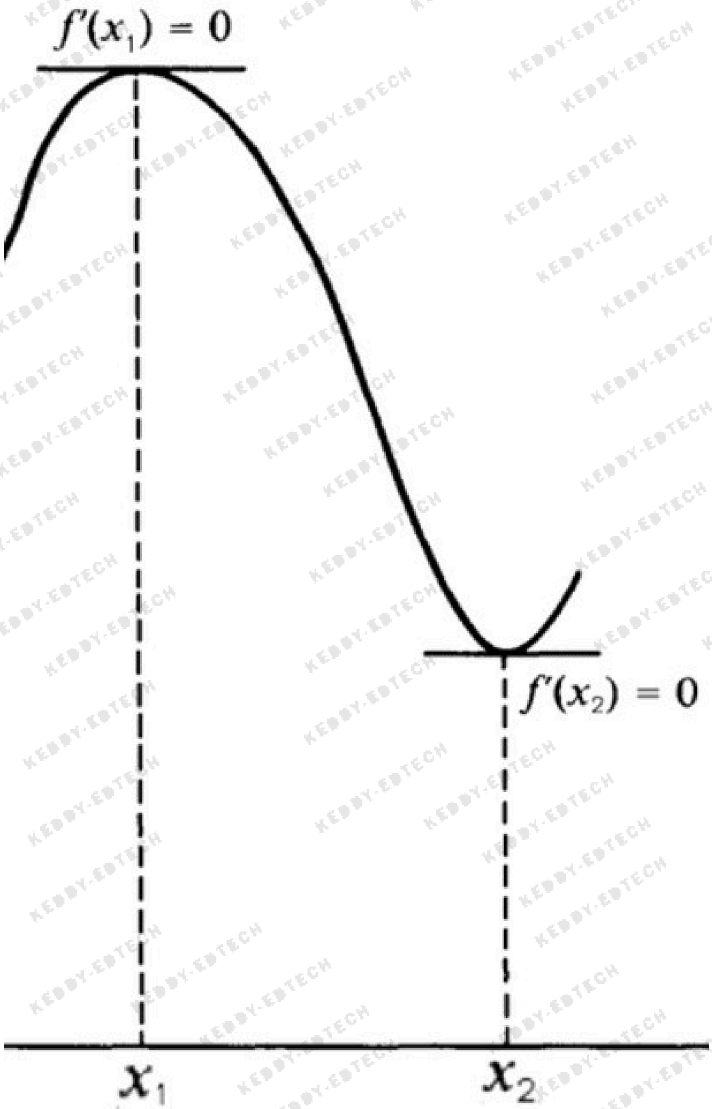
$$f'(x) = (2x-3)^2 - 4$$

The function is increasing when $f'(x) > 0$

$$(2x-3)^2 - 4 > 0 \Rightarrow x > \frac{5}{2} \text{ or } x < \frac{1}{2}$$

1.5. Stationary points

- A stationary, or turning, point is a point on the graph where the gradient is 0.
  - Graphically - The gradient of the tangent, $f'(x)$, is horizontal and parallel to the x-axis at a stationary point.



In this case:

- x_1 represents a local maxima, or a maximum point.
- x_2 represents a local minima, or a minimum point.

First derivative

- The first derivative helps us find the x coordinate of the stationary point.

To find the stationary point:

- Differentiate the function, $f(x)$, to get the first derivative, $f'(x)$.
  - $\frac{d}{dx}f(x) = f'(x)$
- Equate $f'(x)$ to 0, as the gradient of the tangent at any stationary point is always 0. This will give you the x coordinate of the stationary point.
  - $f'(x) = 0$
- Input the value of the x coordinate, or coordinates in some cases, back into the original equation $f(x)$ to get the y coordinate of the stationary point.

Second derivative

- The second derivative helps us find the nature of the stationary point.
  - It helps us figure out if the stationary point is a minima or maxima.

To find the second derivative, differentiate the first derivative.

$$\frac{d}{dx}\left(\frac{dy}{dx}\right) = \frac{d}{dx}f'(x) = f''(x)$$

It is also denoted by

$$\frac{d^2}{dx^2} \text{ or } \frac{d^2y}{dx^2}$$

Inputting the *x* coordinate of the stationary point into the second derivative, *f''*(*x*), will tell us the nature of the stationary point:

Inequality	Nature of stationary point
$\frac{d^2y}{dx^2} > 0$	Minimum point
$\frac{d^2y}{dx^2} < 0$	Maximum point

Example

A curve has equation

$$f(x) = 2x + \frac{2}{x}$$

Find the coordinates of the stationary point on *f*(*x*) and determine its nature.

Answer

To find the coordinate of the stationary point, we find the first derivative and equate it to 0.

$$\frac{d}{dx}(2x + 2x^{-1}) = 2 - 2x^{-2}$$

$$2 - 2x^{-2} = 0 \Rightarrow x = 1$$

Inputting the *x* coordinate back into the original equation, $f(x)$, will give us the *y* coordinate of the stationary point.

$$f(1) = 2(1) + \frac{2}{1} = 3$$

So the coordinate of the stationary point is

$$(1, 3)$$

To find the nature of this stationary point, we find the second derivative by differentiating the first derivative

$$f''(x) = \frac{d}{dx}(2 - 2x^{-2}) = 2 + 4x^{-3}$$

We can now input the *x* coordinate of the stationary point into $f''(x)$

$$f''(1) = 2 + 4(1)^{-3} = 6$$

As $\frac{d^2y}{dx^2} > 0$, this stationary point is a minima.

1.6. Applications of Differentiation

- Differentiation can be used in modelling situations
  - Such as finding the maximum or minimum volume/area a model can have.
- If the variables in an equation vary with time, or any other variable, their rates of change can be connected using

$$\frac{dy}{dx} = \frac{dt}{dx} \times \frac{dy}{dt}$$

Example

The volume, *V*, of a spherical balloon is increasing at a constant rate of 50 cm3/s. Find the rate of increase of the radius when the radius is 10 cm.

Answer

We are given $\frac{dV}{dt} = 50$, and are asked to find the value of $\frac{dr}{dt}$.

To solve the question, we first note that the volume of a sphere is given by

$$V = \frac{4}{3}\pi r^3$$

By differentiating the formula for a volume of a sphere, we get

$$\frac{dV}{dr} = 4\pi r^2$$

So at $r = 10$

$$\frac{dV}{dr} = 4\pi(10)^2 = 400\pi$$

Using the connected rates of change formula, or chain rule:

$$\frac{dV}{dr} = \frac{dt}{dr} \times \frac{dV}{dt}$$

We can substitute in our values

$$400\pi = \frac{dt}{dr} \times 50$$

$$\frac{dt}{dr} = \frac{400\pi}{50} = 8\pi$$

To find the value of $\frac{dr}{dt}$ we can take the reciprocal of the result

$$\frac{dr}{dt} = \frac{1}{8\pi}$$

1. Integration

1.1. Introduction

- The integral can be thought of as the anti-derivative, or the reverse of differentiation.

In mathematical terms

$$\text{If } \frac{d}{dx}F(x) = f(x), \quad \text{then } \int f(x)\,dx = F(x) + C$$

Where:

- c is an arbitrary constant with no specific value.
  - This can be thought of as the value lost due to the differentiation of a constant number.
- dx represents that the function is being integrated with respect to x.

Basics of Indefinite Integration

Reverse power rule

$$\int x^n\,dx = \frac{x^{n+1}}{n+1} + c \quad \text{for any real number } n \neq -1$$

Sum and Difference rule

$$\int f(x) \pm g(x)\,dx = \int f(x)\,dx \pm \int g(x)\,dx$$

Scalar multiple rule

$$\int a \cdot f(x)\,dx = a\int f(x)\,dx \quad \text{where } a \text{ is a constant number}$$

Constant rule

$$\int a\,dx = ax + c \quad \text{where } a \text{ is a constant number}$$

1.2. Reverse Chain Rule

- The reverse chain rule helps us integrate linear functions that are raised to an exponent.

Consider $g(f(x))$ such that

$$\frac{d}{dx} g(f(x)) = g'(f(x)) \times f'(x)$$

We can integrate the result as such

$$\int g'(f(x)) \times f'(x)\,dx = g(f(x)) + c$$

An important application of the reverse chain rule is shown below.
[a and b are constant real numbers]

$$\int (ax+b)^n\,dx = \frac{(ax+b)^{n+1}}{\left[\frac{d}{dx}(ax+b)\right] \times (n+1)} + C$$

This simplifies to

$$\int (ax+b)^n\,dx = \frac{(ax+b)^{n+1}}{a \times (n+1)} + C$$

- This is valid for all real numbers $n \neq -1, 0$

Example

Integrate $\frac{1}{(2x+3)^2}$.

Answer

Using the reverse chain rule

$$\int (2x+3)^{-2} = \frac{(2x+3)^{-2+1}}{\left[\frac{d}{dx}(2x+3)\right] \times (-2+1)} + c$$

This simplifies to

$$-\frac{1}{4x+6} + c$$

1.3. Finding the arbitrary constant

- To be able to find the value of the constant we must be given a point on the curve.

Consider a point (e, f) on the curve $G(x)$, where $G(x) = F(x) + c$.
We are given a derivative $f(x)$. To find the original equation:

$$\int f(x)\,dx = F(x) + c$$

Using the points given

$$f = F(e) + c \Rightarrow c = f - F(e)$$

Example

A curve is such that $\frac{dy}{dx} = 2x^2 - 5$. Given that the point $(3, 8)$ lies on the curve, find the equation of the curve.

Answer

We begin by integrating the derivative given.

$$\int 2x^2 - 5\,dx = \frac{2x^{2+1}}{2+1} - 5x + c$$

Giving us

$$y = \frac{2x^3}{3} - 5x + c$$

We can now substitute in the points given in the question

$$8 = \frac{2(3)^3}{3} - 5(3) + c \Rightarrow c = +5$$

So the equation of the curve is

$$y = \frac{2x^3}{3} - 5x + 5$$

1.4. Definite Integration

- A definite integral is an integral that is described by an upper and lower limit.

Fundamental theorem of Calculus

$$\int_{x=b}^{x=a} f(x)\,dx = [F(x)]_b^a = F(a) - F(b) \quad [\text{Where } b \le x \le a]$$

Where:

- a is the upper limit.
- b is the lower limit.
- Note that adding the arbitrary constant is not necessary in definite integration as it will get cancelled out.

Basics of Definite Integration

- The upper and lower limit must be the same for this rule to apply

Sum and Difference rule

$$\int_b^a f(x) \pm g(x)\,dx = \int_b^a f(x) \pm \int_b^a g(x)$$

Scalar multiple rule

$$\int_b^a zf(x)\,dx = z \times \int_b^a f(x)\,dx \quad [\text{Where } z \text{ is a constant number}]$$

Change of limits rule

$$\int_b^a f(x)\,dx = -\int_a^b f(x)\,dx$$

We also come across some useful results

$$\int_a^a f(x)\,dx = 0$$

$$\int_a^b f(x)\,dx + \int_b^c f(x)\,dx = \int_a^c f(x)\,dx \quad [\text{Where } a < b < c \text{ and } f(x) > 0]$$

Example

Compute the value of $\int_2^4 3x^3 - 4x^2 + 2x + 5 \; dx$.

Answer

We first start by integrating.

$$\int_2^4 3x^3 - 4x^2 + 2x + 5 \; dx = \left(\frac{3x^{3+1}}{3+1} - \frac{4x^{2+1}}{2+1} + \frac{2x^{1+1}}{1+1} + 5x\right)_2^4$$

This simplifies to

$$\left(\frac{3x^4}{4} - \frac{4x^4}{4} + x^2 + 5x\right)_2^4$$

Now we can substitute in the upper and lower limit as such

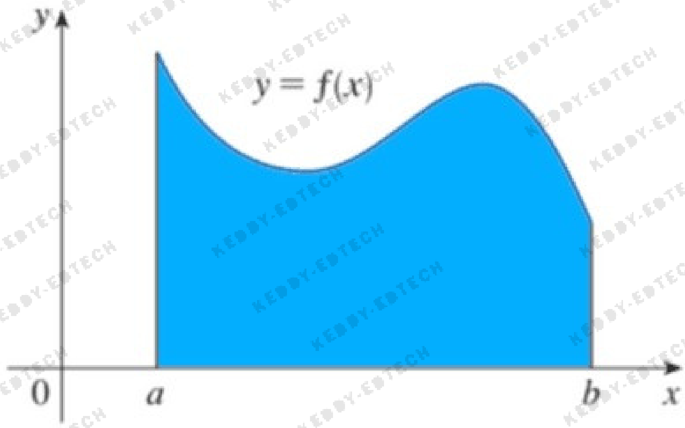
$$\left(\frac{3(4)^4}{4} - \frac{4(4)^4}{4} + (4)^2 + (4)x\right) - \left(\frac{3(2)^4}{4} - \frac{4(2)^4}{4} + (2)^2 + 5(2)\right)$$

$$\left(\frac{428}{3}\right) - \left(\frac{46}{3}\right) = \frac{382}{3}$$

$$\int_2^4 3x^3 - 4x^2 + 2x + 5 \; dx = \frac{382}{3}$$

1.5. Area bounded by the x-axis

Finding the area under a curve is an application of definite integration.



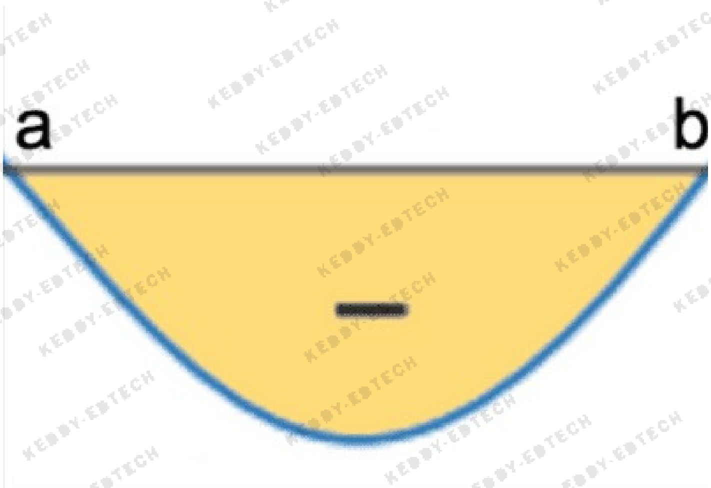
- The definite integral below gives us the area under the graph bounded by the x-axis.

$$\int_{x=a}^{x=b} f(x)dx = F(b) - F(a) \quad [\text{Where } a \le x \le b]$$

- Here $f(x)$ must be in terms of x.
- Integrating a constant, such as $y=c$, with limits a and b will give you the area of a rectangle, bounded by the x-axis, with length c and breadth $b-a$.

The area bounded below the x-axis

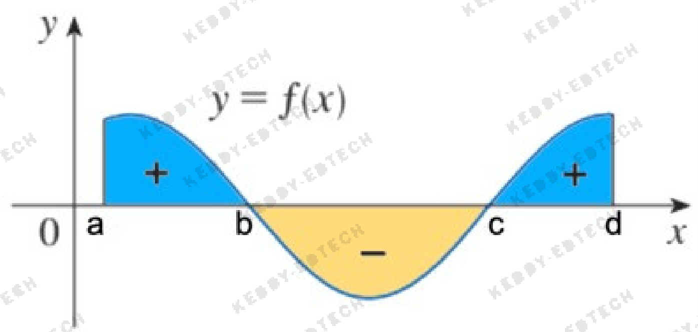
- The area bounded below the x-axis is negative, but the positive value must be taken as area can never have a negative value.



So the area under the x-axis can be written as

$$-\int_a^b f(x)\,dx$$

To find the area under a graph as such:



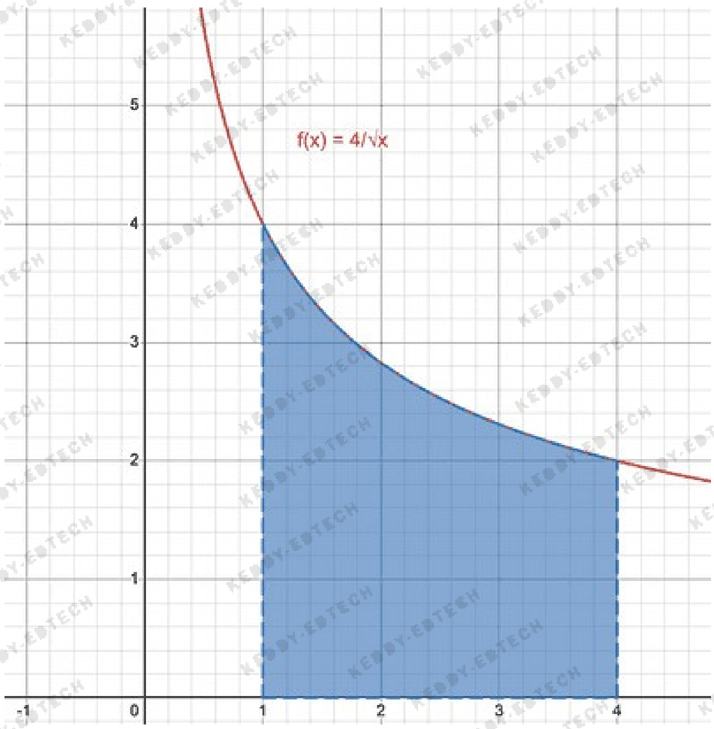
We can use 3 different definite integrals

$$\int_a^b f(x)\,dx - \int_b^c f(x)\,dx + \int_c^d f(x)\,dx$$

Example

A curve has the equation

$$f(x) = \frac{4}{\sqrt{x}}$$



Find the area of the shaded region enclosed by the curve, the x -axis and the lines x=1 and x=4.

Answer

From the question

- x=4 is the upper limit as it is the greater number.
- x=1 is the lower limit as it is the smaller number.

Now we can integrate the function $f(x)$

$$\int_1^4 4x^{-\frac{1}{2}}\,dx$$

Upon integrating, we get

$$\frac{4x^{-\frac{1}{2}+1}}{-\frac{1}{2}+1} = 8x^{\frac{1}{2}}$$

Now we can substitute in our limits

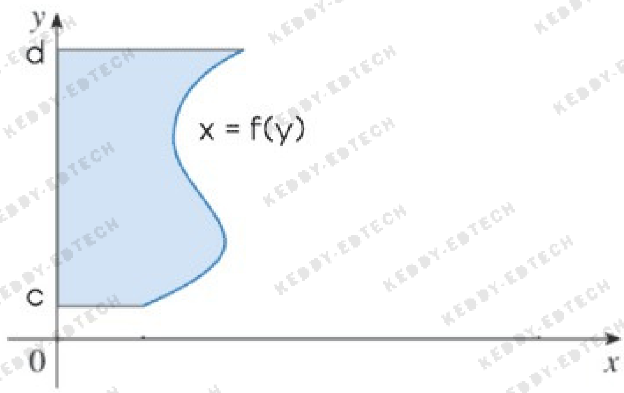
$$[8x^{\frac{1}{2}}]_1^4$$

$$(8(4)^{\frac{1}{2}}) - (8(1)^{\frac{1}{2}}) = 8$$

So the area of the shaded region is 8.

1.6. Area bounded by the y-axis

- To find the area bounded by the y-axis, we need to get an equation in terms of y, such as x=F(y).
  - The limits must converted to y-values, if required.

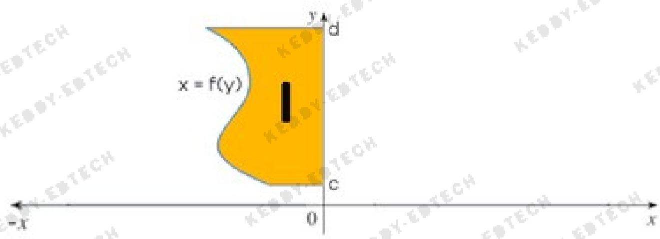


- The definite integral below gives us the area bounded by the y-axis.

$$\int_{y=c}^{y=d} f(y)\,dy = F(d) - F(c) \quad [\text{where } c \le y \le d]$$

Area bounded left of the y-axis

- The area bounded below the y-axis is negative, but the **positive value ** must be taken as area can never have a negative value



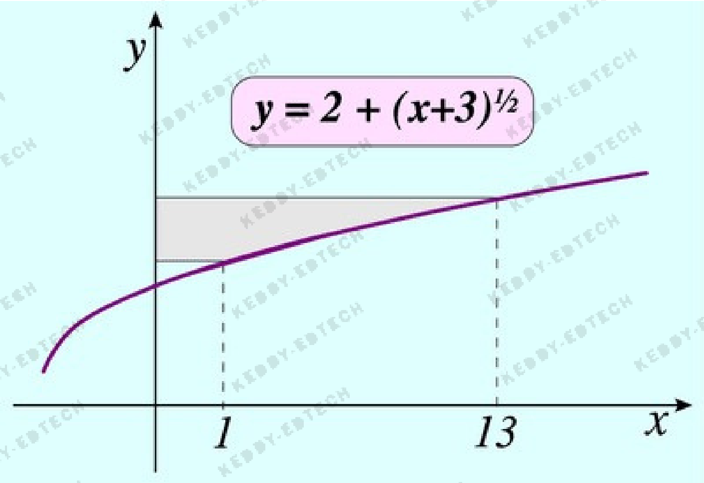
So the area left of the y-axis can be written as

$$-\int_c^d f(y)\,dy$$

Example

A curve has the equation

$$y = 2 + \sqrt{x+3}$$



Find the area of the shaded region, bounded by the y-axis and the curve.

Answer

We start by making x the subject of the formula

$$x=(y-2)^2-3=y^2-4y+1$$

Next, find the corresponding y-values for the x limits.

$$y = 2 + \sqrt{1+3} = 4$$

$$y = 2 + \sqrt{13+3} = 6$$

We can now begin by integrating, along with the upper and lower limit

$$\int_4^6 y^2-4y+1\,dy = \left(\frac{y^3}{3}-\frac{4y^2}{2}+y\right)_4^6$$

Substituting in the limits gives us

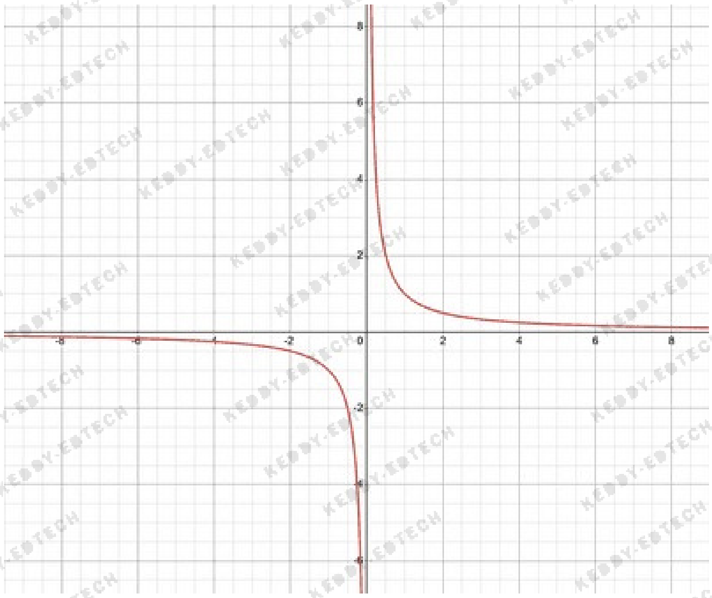
$$\left(\frac{(4)^3}{3}-2(4)^2+4\right)-\left(\frac{(6)^3}{3}-2(6)^2+6\right)=\frac{38}{3}$$

So the area of the shaded region is $\frac{38}{3}$.

1.7. Improper Integration

- Improper integration is a type of definite integration where the function being integrated is undefined at the limits given.

Consider the graph of the function $f(x)=\frac{1}{x}$



Here:

- As x approaches 0, $f(x)$ approaches ∞
  - This is written as $x\to0$, $f(x)\to\infty$
- As x approaches ∞, $f(x)$ approaches 0
  - This is written as $x\to\infty$, $f(x)\to0$
- This is true for the negative x and y axis as well.

Example

The function f is defined by

$$\frac{1}{(4x+2)^2},\quad x > -\frac{1}{2}$$

Find $\int_1^\infty f(x)\,dx$.

Answer

We begin by integrating

$$\int_1^\infty (4x+2)^{-2} = \left(\frac{(4x+2)^{-2+1}}{4(-2+1)}\right)_1^\infty$$

Substitute a dummy variable, t, for the upper limit and evaluate the definite integral

$$\left(\frac{-1}{16t+8}\right)-\left(\frac{-1}{16(1)+8}\right)$$

As $t\to\infty$, $\frac{-1}{16t+8}\to 0$.

$$0+\frac{1}{24}$$

So the answer to the improper integral is $\frac{1}{24}$.

Example 2

Evaluate whether the integral $\int_0^2 \frac{5}{x^2}\,dx$ is defined or undefined.

Answer

We begin by integrating

$$\int_0^2 5x^{-2}\,dx = \left(\frac{5x^{-2+1}}{-2+1}\right)_0^2$$

Substitute a dummy variable, *t*, for the lower limit and evaluate the definite integral

$$\left(\frac{-5}{2}\right) - \left(-\frac{5}{t}\right)$$

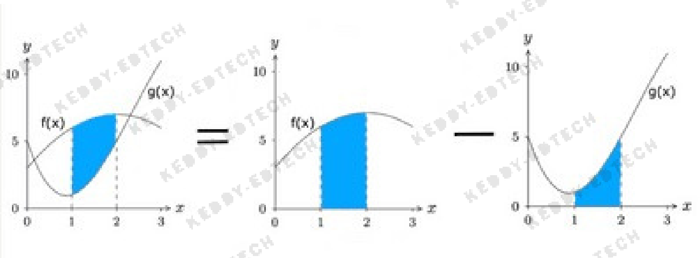
As $t \to 0$, $-\frac{5}{t} \to \infty$.

$$\frac{-5}{2} + \infty$$

As there is no finite value given to this improper integral, it is undefined.

1.8. Area between curves

- The area between two curves can be found by finding the difference of the areas of the top curve and bottom curve.



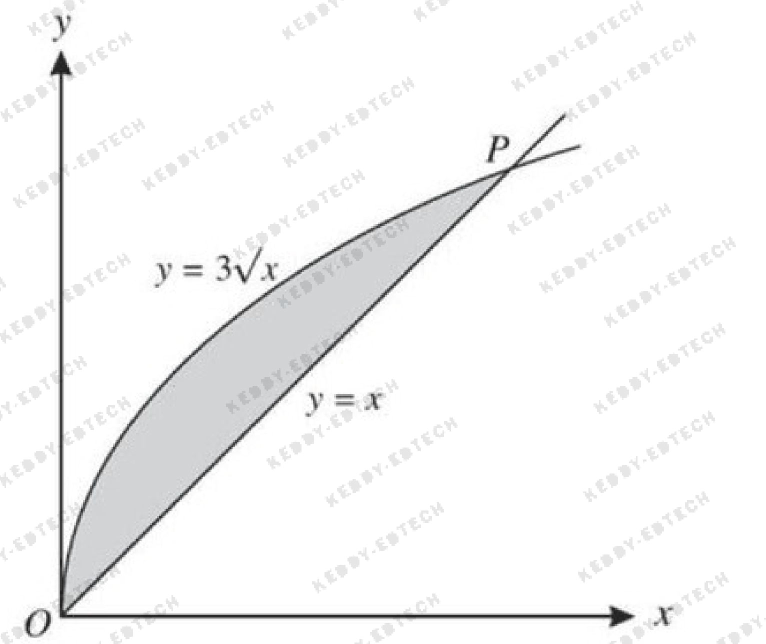
We can find the area between two curves as such

$$\int_a^b f(x) - g(x)\,dx \quad \text{[Where } a \le x \le b\text{]}$$

- While finding the area between two curves, it is important to identify which function is on top and on the bottom.
  - You may also need to find their points of intersection, and use them as the upper and lower limit in questions.

Example

The curve $y = 3\sqrt{x}$ and the line $y = x$ intersect at the origin *O* and the point *P* .



Find the area of the shaded region.

Answer

We must first find the points of intersection, to use them as our limits. The point *O* has coordinates (0,0). To find the coordinate of the point *P*, we must equate both curves.

$$3\sqrt{x} = x \Rightarrow x = 9$$

The point *P* has coordinates (9,9).

Method 1

Next we notice that curve $y = 3\sqrt{x}$ is above the line $y = x$. So we subtract the area of the line from the curve.

$$\int_0^9 3\sqrt{x}\,dx - \int_0^9 x\,dx = \int_0^9 3\sqrt{x} - x\,dx$$

$$\left(2x^{\frac{3}{2}} - \frac{x^2}{2}\right)_0^9$$

$$0 - \left(2(9)^{\frac{3}{2}} - \frac{9^2}{2}\right) = 13.5$$

The area between the two curves is 13.5.

Method 2

Next we notice that curve $y = 3\sqrt{x}$ is above the line $y = x$. So we subtract the area of the line from the curve. The line is a triangle with base length 9 and height 9.

$$\int_0^9 3\sqrt{x}\,dx - \left(\frac{1}{2} 9 \times 9\right)$$

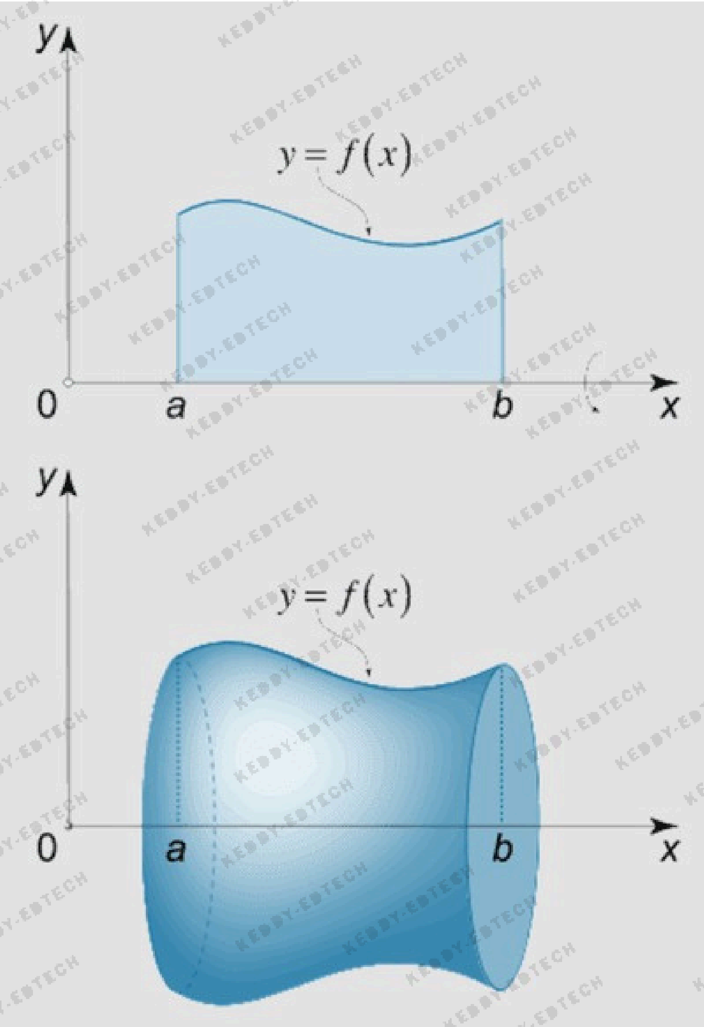
$$\left(2x^{\frac{3}{2}}\right)_0^9 - \left(\frac{81}{2}\right) = 2(9)^{\frac{3}{2}} - \frac{81}{2} = 13.5$$

The area between the two curves is 13.5.

1.9. Volume of Revolution

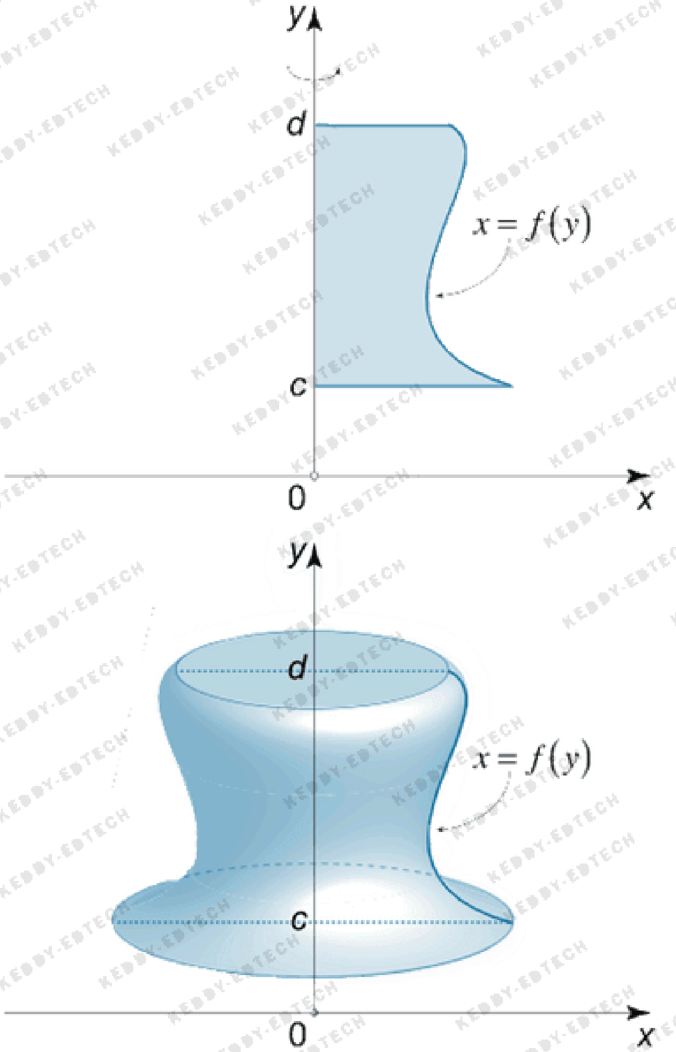
- The volume of revolution of a solid rotated around an axis by 360∘ degrees gives us a 3D shape.
- It is also possible to find the volume between two solid areas by subtracting the upper volume from the lower volume.
- To find the volume of revolution around the *x*-axis by 360 degrees

$$V = \pi \int_{x=a}^{x=b} y^2 dx$$



- To find the volume of revolution around the *y*-axis by 360 degrees

$$V = \pi \int_{y=c}^{y=d} x^2 dy$$



- Note that you do not need to square the equation if it is already in terms of *y*2 or *x*2

Example 1
Find the volume of the solid of revolution formed when the area under the graph of *y*=*x*3 between x-coordinates 3 and 6 rotates 360° around the x-axis

Answer
First we need to get the equation in terms of *y*2

$$y=x3 \Rightarrow y2=x6$$

Now we can evaluate the integral using the limits given

$$V = \pi \int_{3}^{6} x^6 dx$$

$$\pi \left(\frac{x^{6+1}}{6+1} \right)_{3}^{6}$$

$$V = \pi \left(\frac{6^7}{7} - \frac{3^7}{7} \right)$$

$$V = \frac{277749\pi}{7}$$

So the volume of revolution the around x-axis by 360 degrees is ~~277~~ 749π

Example 2

Find the volume of revolution when the area under the graph of $y4=x2$ between y-coordinates 2 and 4 rotated 360° around the y-axis.

Answer

As the equation is already in terms of $x2$, we can directly evaluate the integral.

$$V = \pi \int_{2}^{4} y^4 dy$$

$$\pi \left(\frac{y^{4+1}}{4+1} \right)_{2}^{4}$$

$$\pi \left(\frac{4^5}{5} - \frac{2^5}{5} \right)$$

$$V = \frac{992\pi}{5}$$

So the volume is equal to $\frac{992\pi}{5}$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